NOTES ON

WAVELET TRANSFORM AND ITS ANALYSIS
OF RANDOM FIELDS

Prepared by Le Thai Hoa

2005
WAVELET TRANSFORM AND ITS ANALYSIS
OF RANDOM FIELDS

1. Introduction

(1) In practice, the most of the signals or processes are represented under a form of time-domain ones (function of time) as native and raw formats. Time-domain signals are plotted by time axis and amplitude one (as time-variant variable), then we obtain the time-amplitude representation. Furthermore, signals always contain frequency-dependant components as nature of signals. This frequency information lets us know which frequency components are existing on raw signal and how contributions of such frequency components participate on signal (corresponding to any oscillation), in this case we obtain the frequency-amplitude representation. However, it is very significant to express the signal under a form of the time-frequency-amplitude representation.

(2) Mathematical transformations are applied for the original time-domain signals to obtain further and hidden information inside the raw signals. Transformations between time- and frequency-domain of signals play important role. Some of the most applicable mathematical transformations in scientific and engineering applications will be mentioning as follows:

i) Hilbert transform (Time-domain representation): System identification

ii) Fourier Transform (Frequency-domain representation) /Laplace Transform (Nondimensionless Laplace-domain representation) /Z Transform (Frequency-domain representation of discrete signals)

iii) Wavelet Transform (Time-domain representation)

(3) Signal classification of engineering application can be categorized by i) Deterministic signals and ii) Random (stochastic) signals, in which the former can
be expressed by explicit function of time and variables, whereas the later only can expressed functions of probability-based variables. Almost signals in practice are random. However, the random signals can be determined by the statistic-based characteristics as follows: 

1) Amplitude distribution as mean value/expectation and r.m.s/standard deviation,
2) Auto- and cross-correlation functions and
3) Power spectral density (PSD) as power contribution of each frequency components on signal.

(4) Classification of signals can be given by such following diagram:

Note:

1) Random stationary signals are that their mean value and correlation of discrete signals do not vary on time, whereas non-stationary signals are whose their mean value and correlation are time-variant.

2) Almost signals are non-stationary, but narrow time-interval segments of non-stationary signal are assumed and accepted as stationary one.
3) Random stationary signals can be decomposed into harmonic sinusoidal waves thanks to the Fourier Series Decomposition.

(5) Transformations between time- and frequency-domain quantities like the Fourier Transform (FT) can be applied only for stationary signals, but can not for non-stationary one, because frequency components of the non-stationary signal depend on the time. By inclusion of the window functions, however, the Fourier Transform can be applied for the non-stationary signals (as assumption that narrow-time segments of non-stationary signal is stationary one), that known as the Short Time Fourier Transform (STFT). Wavelet Transform (WT) can treat with non-stationary signals.

(6) In the time-domain representation (time-amplitude) as raw format of signals, there is no any information on frequency, on the contrary, in the frequency-domain representation (frequency-amplitude), no time information is available. The question come in mind that how to obtain both the time and the frequency information. This question can be solved thanks to the Wavelet Transform under so-called the time-frequency representation (time-frequency-amplitude).

(7) For a short, the Wavelet Transform is advantageous out of Fourier Transform: i) Time and frequency information of signals can be revealed, ii) Stationary and non-stationary signals can be treated.

(8) In this study, applications of the Wavelet Transform in the Wind Engineering will concentrate on decomposition and reconstruction of velocity and pressure fields, as well as relationship modelling between ongoing velocity field and wind-induced surface pressure field.
2. **Fourier series and harmonic representation of periodic signals**

(1) Fourier Transform and Fourier Series are separated to treat with two different kinds of functions: periodic and . Periodic signals with T-period can be decomposed by a sum of harmonic functions thanks to the Fourier Series as as follow:

\[
x(t) = x(t + T)
\]

\[
x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} \left[ a_m \cos(m\omega_0 t) + b_m \sin(m\omega_0 t) \right]
\] (Type 1)

\[
x(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty} c_m \cos(m\omega_0 t + \theta_m)
\] (Type 2)

\[\omega_0: \text{Fundamental frequency (rad/s), } \omega_0 = \frac{2\pi}{f_o} = 2\pi T\]

\[m: \text{Times of fundamental frequency}
\]

\[(m=1: \text{fundamental harmonic term with } \omega_0, \ m=2: \text{2}\text{nd harmonic term with } 2\omega_0, \ m\text{th harmonic term with } m\omega_0)\]

\[a_0, a_m, b_m, c_m: \text{Fourier coefficients of series (} a_m, b_m: \text{real; } c_m: \text{complex)}\]

(2) The Fourier series representation is provided under complex form:

\[
x(t) = \sum_{m=-\infty}^{\infty} \alpha_m e^{j m \omega_0 t}
\]

\[
\alpha_m = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-j m \omega_0 t} dt
\]

(3) Fourier coefficients

\[
a_0 = \frac{2}{T} \int_{0}^{T} p(t) dt
\]
\[ a_m = \frac{2}{T} \int_0^T p(t) \cos(m\omega_0 t) dt \]

\[ b_m = \frac{2}{T} \int_0^T p(t) \sin(m\omega_0 t) dt \]

\[ |c_m| = \sqrt{a_m^2 + b_m^2} ; \quad c_m = |c_m| e^{j\theta_m} \]

\[ \theta_m = \tan^{-1}\left( \frac{b_m}{a_m} \right) \]

(4) Role of Fourier Series: Fourier Series decomposes a signals to sums of harmonic functions of different frequencies. This Fourier Series is applied on signal filtering procedure (consists of: Decomposition and Filtering).

3. Fourier Transform (FT)

(1) Fourier Transform (Kinchint-Weiner’s pair) can be expressed as follows:

\[ X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = |X(\omega)| e^{j\phi(\omega)} \]

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \]

(Inverse Fourier Transform or Signal Reconstruction)

\[ X(\omega) = \langle x(t), e^{-j\omega t} \rangle : \text{so-called ‘Inner product’} \]

(2) Fundamental characteristics of FT

1) **Time shifting (Time lag):** Function x(t) be shifted by time lag \( \tau \)
\[
x_\tau(t) = x(t - \tau)
\]

then
\[
X_\tau(\omega) = X(\omega)e^{-j\omega \tau}
\]

2) **Time scaling**: Time \( t \) be scaled by \( a: \text{scale parameter} \)
\[
x_a(t) = x(at)
\]

then
\[
X_a(\omega) = \frac{1}{|a|}X\left(\frac{\omega}{a}\right)
\]

3) **Frequency shifting**
\[
X_{\psi}(\omega) = X(\omega - \omega_0)
\]

then
\[
x_{\psi}(t) = x(t)e^{j\omega_0 t}
\]

4) **Frequency scaling**
\[
X_a(\omega) = X(a\omega)
\]

then
\[
x_a(t) = \frac{1}{a}x\left(\frac{t}{a}\right)
\]

5) **Parseval’s Theorem**
\[
\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega
\]

(3) **Role of FT and explanation:**

1) Random stationary signals can be represented as sums of sine wave functions thanks to FT. If signal has time-variant frequency (as non-stationary signal), FT will be invalid. Therefore, it is important to make sure whether signal is stationary or not, prior to processing by FT.

2) It can be explained the Fourier Transform as follows: The signal \( x(t) \) is multiplied with exponential term \( (e^{-j2\pi f}) \) at some certain frequency \( f \), then integrated over ‘time range’. Noting that the exponential term decomposed
as real and imaginary harmonic components: $e^{-j2\pi ft} = \cos(2\pi ft) - j\sin(2\pi ft)$. If the original signal contains an amplitude component of certain frequency $f$, the signal and harmonic component coincide, then their product will be relatively large value. On the contrast, if signal do not contain component of frequency $f$, then their product will be zero. In case, signal has amplitude component of frequency $f$, however this component of frequency does not dominate on original signal, then product will give relatively small value. Integration of product over all time range, it also means that products will be calculated with every value of frequency.

3) The Fourier Transform reveals that whether or not component of certain frequency exits in original signals, however, FT can not know In-What-Times these frequency components occur.

4. Short Time Fourier Transform (STFT) and Gabor Transform

(1) As above-mentioned, FT has some shortcomings: i) can not process with non-stationary signals, ii) can not reveal where in time dominant component of frequency occurs. However, it is assumed that some narrowed segments of non-stationary signal are stationary. Picking up these segments has been carried out by a so-called ‘Window functions’.

(2) The Window function $w(t)$ is defined that product of $x_\tau(t) = x(t)w(t-\tau)$ as:

$$x_\tau(t) = \begin{cases} x(t), & t \in [\tau - \Delta\tau, \tau + \Delta\tau] \\ 0, & otherwise \end{cases}$$

(3) STFT can be expressed similar to FT as follows:

$$X(\tau, \omega) = \int_{-\infty}^{\infty} x(t)w^*(t-\tau)e^{-j\omega t} dt$$
Inverse Short Time Fourier Transform (ISTFT)

\[ x_\tau(t) = x(t)w(t - \tau) = \int_{-\infty}^{\infty} X(\tau, \omega)e^{j\omega t} d\omega \]

\[ x(t) = \frac{1}{2\pi|w(t)|} \int_{-\infty}^{\infty} X(\tau, \omega)w^*(t - \tau)e^{j\omega t} d\omega d\tau \]

(4) Gabor Transform as STFT with Gabor’s window function like Gaussian function

\[ w_\sigma^g(t) = \frac{1}{2\pi\sigma} e^{-\frac{t^2}{4\sigma}} \]

(5) Some explanations of STFT:

i) STFT \( X(\tau, \omega) \) contains two parameters: time lag \( \tau \) and frequency \( \omega \), thus STFT is considered as time-frequency representation.

ii) The window function can be shifted by parameter \( \tau \), but window length is finite. This means that window function only cope with a segment of signal, which cause the frequency solution to get poorer. On the other hand, we no longer know the exact frequency components but only know a frequency band that components of frequencies exits. This is main shortcoming of STFT.

5. Continuous Wavelet Transform (CWT)

(1) Differing from the Short Time Fourier Transform (STFT), the window function is replaced by the wavelet function \( \psi(t) \) (or the mother wavelet) in the Wavelet Transform. Characteristics of the wavelet functions are follows:

i) Oscillatory function

ii) Fast decay toward zero
Note: Thanks to conditions i) and ii) that ‘Wavelet’ means small wave

iii) Negative and positive values

iv) Admissibility condition: Mother wavelet is scaled (or dilated) by scale parameter $s$ and translated (or shifted) by translation parameter $\tau$

$$\psi_{s,\tau}(t) = \frac{1}{\sqrt{s}} \psi \left( \frac{t - \tau}{s} \right)$$

Above relationship is called ‘Unitary affine mapping’ or ‘relationship between mother wavelet and daughter wavelet’

v) Orthonormality condition

$$\int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1$$

vi) Zero-mean value condition

$$\int_{-\infty}^{\infty} \psi(t) dt = 0$$

(2) There are many mother wavelet functions and their relatives invented and can be divided by two groups: i) Discrete wavelet functions and ii) Continuous wavelet functions. Discrete wavelets: 1) Haar wavelet, 2) Daubechies wavelet and its family…Continuous wavelets: 1) Morlet wavelet, 2) Mexican Hat wavelet, 3) Mayer wavelet, 4) Hermitian wavelet…

(3) Parameters of mother wavelets

i) Scale $s$ (Inverse of frequency) or window lengths: High scale (non-detailed global view of signal) corresponds to low frequency, whereas low scale
(detailed local view of signal) corresponds to high frequency. This means that low frequency (high scale) corresponds to a global information of a signal, whereas high frequency (low scale) corresponds to a detailed information of hidden pattern in the signal. Scale as a mathematic operation implies to dilate or compress of signal. Larger scale corresponds to dilate signal and smaller scale implies to compress signal. Scale provides information of frequency.

ii) Translation \( \tau \) (Time interval) or window locations: With translation parameter, the window shifts through the signal. Translation provides information of time.

(4) Continuous Wavelet Transform:

\[
\Psi(\tau, s) = \int_{-\infty}^{\infty} x(t) \frac{1}{\sqrt{s}} \psi^*(\frac{t-\tau}{s}) dt 
\]

\( \Psi(\tau, s) \): Wavelet-domain coefficient (Time-frequency coefficient) depending on scale \( s \) and translation \( \tau \)

Reconstructing an original signal thanks to Inverse Wavelet Transform

\[
x(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Psi(\tau, s) \frac{1}{s^2} \psi(\frac{t-\tau}{s}) ds d\tau 
\]

\( C_\psi \): Admissibility constant (or normalized constant)

\[
C_\psi = \int_{-\infty}^{\infty} \left| \hat{\psi}(\omega) \right|^2 d\omega 
\]

\[
\hat{\psi}(\omega) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{s}} \psi(\frac{t-\tau}{s}) e^{-j\omega t} dt 
\]
(5) Morlet wavelet (after Morlet 1984)

The Morlet wavelet was formulated as product of exponential term and harmonic one:

\[
\Psi(t) = e^{\frac{-t^2}{2}} [\cos \omega_0 t + j \sin \omega_0 t] = e^{\frac{-t^2}{2}} e^{j \omega_0 t} = e^{\frac{-t^2}{2} + j \omega_0 t}
\]

As can be seen that the Morlet wavelet is Gaussian-windowed Fourier Transform with sine and cosine oscillation at central frequency \( f_0 = \omega_0 / 2\pi \). Fourier Transform of the Morlet wavelet is:

\[
\Psi(sf) = \sqrt{2\pi} e^{-\pi^2 (sf - f_0)}
\]

(6) Mexican hat wavelet

Mexican wavelet is the normalized, second derivative of a Gaussian function (known as a special case of Hermittian wavelet family)

Gaussian function:

\[
G(t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{t^2}{2\sigma^2}}
\]

Mexican hat wavelet:

\[
\Psi(t) = \frac{1}{\sqrt{2\pi\sigma^3}} e^{-\frac{t^2}{2\sigma^2}} \left( \frac{t^2}{\sigma^2} - 1 \right)
\]
(7) Energy distribution of Wavelet Transform

Time-domain energy of signals

\[ E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \]

Frequency-domain energy of signals

\[ E_{X(\omega)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega \]

Time-frequency domain energy of signals

\[ E_{\Psi(\tau, s)} = \frac{1}{C_{\Psi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{s^2} |\Psi(\tau, s)|^2 \, dsd\tau \]

Differential energy of in a differential area of scale-translation plane in wavelet domain is defined as Scalogram:

\[ Sca\, log\, ram = \frac{1}{s^2} |\Psi(\tau, s)|^2 \, d\tau ds \]

6. Discrete Wavelet Transform (DWT)

(1) Discrete Wavelet Transform (DWT) also refers to Wavelet Transform in which i) Mother wavelet is discrete function, or ii) Mother wavelet is sampled discretely.

Furthermore, DWT provides sufficient information for analysis and synthesis of original signal with more effective computation than CWT.
(2) CWT was computed by scaling (dilation or compression) of analysis window of mother wavelet, then shifting the window in time, then multiplying with original signal and integrating over all time. In case of DWT, the filters of different cutoff frequencies are used to analyze signal at different scales, then signal is passed through a series of high-pass filters to process high-frequency components, then through low-pass filters to process low-frequency components.

(3) CWT is widely used for signal analysis, whereas DWT is mostly applied for signal coding.

(4) Haar wavelet (after Haar 1909 known as D2 wavelet of Daubechies wavelet family)

The Haar Wavelet (the simplest wavelet) can also be described as a step function with

\[
f(x) = \begin{cases} 
1 & 0 \leq x < 1/2, \\
-1 & 1/2 \leq x < 1, \\
0 & \text{otherwise}.
\end{cases}
\]
Le Thai Hoa – Wavelet transform and analysis of random fields
Haar wavelet

Morlet wavelet function

Morlet wavelet
Mexican Hat wavelet function

Meyer scaling function

Meyer wavelet function
Daubechies 2 wavelet function

Daubechies 10 wavelet function
Wavelet Transform using mother Morlet wavelet
Wavelet Transform using mother Mexican Hat wavelet
LETHAI HOA – WAVELET TRANSFORM AND ANALYSIS OF RANDOM FIELDS

Signal

Amplitude

Time

30ms 80ms

Fourier Transform

Magnitude

Frequency

0Hz - 50Hz

Wavelet coefficients

Scales

Translation

0Hz - 50Hz
2D time-frequency representation of impulse signal

3D time-frequency representation of impulse signal
Rectangular impulse signal

Fourier Transform

Rectangular impulse signal
Wavelet coefficients
Wavelet coefficients

Scale
Translation
Le Thai Hoa – Wavelet transform and analysis of random fields
Wavelet coefficients

Scales

Translation

COEFS

0

0.1

0.2

0.3

9.7 1.85 7.93 6.76 2.5 4.9 3.7 1.3 2.5 1.9 13 7 1

20

40

60

80

100

Le Thai Hoa – Wavelet transform and analysis of random fields