Time and frequency domain gust response of bridges using proper orthogonal decomposition

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ABSTRACT: Gust response prediction of bridges burdens difficulties due to a projection of spatially-correlated turbulent loading on the generalized structural coordinates. Proper orthogonal decomposition is used to decouple the turbulent loading into the orthogonally turbulent modes which then are associated with the structural modes. This paper will present applications of proper orthogonal decomposition in decoupling and simulating of the turbulent loading, then gust response of bridges are formulated in both the time and frequency domains using the modal and proper transformations.

KEYWORDS: gust response, proper orthogonal decomposition; modal transformations; turbulence field; simulation; frequency and time domain analyses

1 INTRODUCTION

Gust response prediction of bridges subjected to the turbulent-induced forces proposed by Davenport, 1963 4) in the frequency domain. The time domain analysis has been developed by some authors 2) 8). As a principle, MDOF motion equations of bridges were decoupled into generalized coordinates and orthogonally structural modes using the modal transformation. However, the inevitably difficulty is to decompose the spatially-correlated turbulent loading, which then are associated with generalized structural coordinates. In the conventional approaches, the joint acceptance function technique has been used 2) 4).

The proper orthogonal decomposition (POD), known as the Karhunen-Loeve decomposition, was firstly applied for random turbulent fields by Lumley, 1970 7) as a stochastic decomposition to decouple multi-variate random turbulent fields. POD also has been used for the simulation of turbulent field 3) 5), the pressure field analysis 12) and the physical phenomenon identification 8). The proper transformations have been branched by either covariance or spectral proper transformations, which depends on basic matrix formed from either covariance or cross spectral matrices. New approach for dynamic and gust response analysis of structures, so-called double modal transformation, has been proposed recently by Carassale et al.,1999 1); Solari & Carassale, 2000 10) by which the structural modes can be associated with the turbulent modes decomposed by the proper transformations. The spectral-based double modal transformation has been applied for gust response prediction of tall building 3), and for that of bridge 11). The gust response prediction of bridges, however, needs to account the frequency-dependant correction function such as an aerodynamic admittance. The time domain gust response, furthermore, using the covariance proper transformation is promising because of its comprehensive solutions for nonlinear and unsteady aerodynamics.

In this paper, the POD-based covariance and spectral proper transformations will be presented with emphasis on their applications to decouple and simulate the spatially-correlated turbulent loading and formulation of gust response prediction of bridges in both the time and frequency domains. Numerical example of cable-stayed bridge is taken for investigation.
2 SPATIALLY-CORRELATED TURBULENT FIELD

Turbulent field acting on bridge can be represented as multi-variate zero-mean Gaussian random processes of velocity fluctuations \( u(t), w(t) \) at \( N \) structural discrete nodes (Figure 1):

\[
\begin{align*}
\mathbf{u}(t) = [u_1(t), u_2(t), \ldots, u_N(t)]^T; \quad \mathbf{w}(t) = [w_1(t), w_2(t), \ldots, w_N(t)]^T
\end{align*}
\]

Statistical quantities such as covariance and power spectrum are commonly used for representing random processes in the time and frequency domains. Spatially-correlated turbulent field can be characterized by squared covariance, cross spectral matrices defined as:

\[
\begin{align*}
C_v = [R_{u(t),u(t)}(0)]_{N \times N}; \quad S_v = [S_{u(t),u(t)}(n)]_{N \times N}; \quad m,k = 1,2,\ldots,N
\end{align*}
\]

where \( v \) denotes \( u(t) \) or \( w(t) \); \( C_v, S_v \) : covariance and cross spectral matrices; \( R_{u(t),u(t)}(0), S_{u(t),u(t)}(n) \) : zero-time-lag covariance and cross spectral elements between \( u_m(t) \) and \( u_k(t) \) at nodes \( m, k \), which are determined as follows:

\[
R_{u(t),u(t)}(0) = E[u_m(t)u_k^T(t)]; \quad S_{u(t),u(t)}(n) = \sqrt{S_{u(t),u(t)}(n)S_{u(t),u(t)}(n)coh_{u}(n,\Delta mk)}
\]

where \( E[.] \) denotes the expectation operator; \( S_{u(t),u(t)}(n), S_{u(t),u(t)}(n) \) : auto spectra of \( u(t) \) at nodes \( m, k \); \( coh_{u}(n,\Delta mk) \) : spatial coherence function between nodes \( m, k \).

The uniform turbulent-induced forces (buffeting forces) are expressed due to the quasi-steady theory corrected by frequency-dependant admittance functions, see Figure 2:

\[
\begin{align*}
L_h(t) &= \frac{1}{2} \rho U^2 B[L_cX_{L_{uu}}(n)\frac{2u(t)}{U} + (C_L + C_D)X_{L_{uw}}(n)\frac{w(t)}{U}] \quad (4a) \\
D_h(t) &= \frac{1}{2} \rho U^2 B[L_cX_{D_{uu}}(n)\frac{2u(t)}{U} + (C_D - C_L)X_{D_{uw}}(n)\frac{w(t)}{U}] \quad (4b) \\
M_h(t) &= \frac{1}{2} \rho U^2 B^2[L_cX_{M_{uu}}(n)\frac{2u(t)}{U} + C_M X_{M_{uw}}(n)\frac{w(t)}{U}] \quad (4c)
\end{align*}
\]

where \( L_c, C_D, C_M \) : aerodynamic coefficients at balanced angle; \( C_L, C_D, C_M \) : First-order derivatives with respect to angle of attack; \( X_{F_{\alpha}}(F = L, D, M; \alpha = u, w) \) : aerodynamic transfer functions between turbulent components and forces. Accordingly, full-scale buffeting forces can be obtained due to a linearized force distribution \([1]\):

\[
F_h(t) = \frac{1}{2} \rho UB[L_cX_{F_{\alpha}}(n)u(t) + C_uX_{F_{\alpha}}(n)w(t)]
\]

where \( C_u, C_w \) : full-scale force coefficient matrices.

3 PROPER ORTHOGONAL DECOMPOSITION

The main idea of the POD is to find out a set of orthonormal basic vectors which can expand a multi-variate random process into a sum of products of these basic orthogonal vectors and
single-variant uncorrelated random processes. POD-based proper transformations are summarized as follows 2)10)11):

### 3.1 Covariance matrix-based formulation and its covariance proper transformation

The covariance matrix-based orthogonal vectors are found as the eigenvector solutions of the zero-time-lag covariance matrix $C_\nu(n)$ of the N-variate random turbulent field $\nu(t)$:

$$C_\nu(n) = \Gamma_\nu \Theta_\nu$$  \hspace{1cm} (6)

where $\Theta_\nu = [\theta_{1,\nu}, \theta_{2,\nu}, ..., \theta_{N,\nu}]$, $\Gamma_\nu = \text{diag}(\gamma_{1,\nu}, \gamma_{2,\nu}, ..., \gamma_{N,\nu})$: covariance eigenvalue and eigenvector (turbulent mode) matrices, which satisfy the orthonormal conditions:

$$\Theta_\nu \Theta_\nu^T = I; \quad \Theta_\nu C_\nu \Theta_\nu^T = \Gamma_\nu$$  \hspace{1cm} (7)

Accordingly, the turbulence field and its covariance matrix can be expressed due to the covariance proper transformation as summed approximation as follows:

$$\nu(t) = \Theta_\nu \tilde{x}_\nu(t) \approx \sum_{j=1}^{M} \theta_{j,\nu} \tilde{x}_{\nu,j}(t) \quad ; \quad C_\nu = \Theta_\nu \Gamma_\nu \Theta_\nu^T \approx \sum_{j=1}^{M} \theta_{j,\nu} \gamma_{j,\nu} \theta_{j,\nu}^T$$  \hspace{1cm} (8)

where $\tilde{x}_\nu(t) = \{\tilde{x}_{\nu,1}(t), \tilde{x}_{\nu,2}(t), ..., \tilde{x}_{\nu,N}(t)\}^T$: covariance-based turbulent principal coordinates as the N-variate uncorrelated Gaussian random process; $M$: number of truncated turbulent modes ($M < N$). Covariance principal coordinates can be determined from observed data:

$$\tilde{x}_\nu(t) = \Theta_\nu^{-1} \nu(t) \quad ; \quad \nu(t) \Theta_\nu = \sum_{j=1}^{N} \nu_j(t) \theta_{j,\nu}$$  \hspace{1cm} (9)

### 3.2 Spectral matrix-based formulation and its spectral proper transformation

Similarly, the spectral eigenvalues and eigenvectors are found based on the eigen problem of the cross spectral matrix $S_\nu(n)$ of random turbulent process $\nu(t)$:

$$S_\nu(n) \Psi_\nu(n) = \Lambda_\nu(n) \Psi_\nu(n)$$  \hspace{1cm} (10)

where $\Lambda_\nu(n) = \text{diag}(\lambda_{1,\nu}(n), \lambda_{2,\nu}(n), ..., \lambda_{N,\nu}(n))$, $\Psi_\nu(n) = [\psi_{\nu,1}(n), \psi_{\nu,2}(n), ..., \psi_{\nu,N}(n)]$: spectral eigenvalue and eigenvector (turbulent mode) matrices, which also satisfy the orthonormal conditions:

$$\Psi_\nu^T(n) \Psi_\nu(n) = I; \quad \Psi_\nu^T(n) S_\nu(n) \Psi_\nu(n) = \Lambda_\nu(n)$$  \hspace{1cm} (11)

Thus, the Fourier transform and the cross spectral density matrix $S_\nu(n)$ can be represented under the spectral proper transformation as follows:

$$\hat{\nu}(n) = \Psi_\nu(n) \tilde{\nu}_\nu(n) = \sum_{j=1}^{M} \psi_{\nu,j}(n) \tilde{\nu}_{\nu,j}(n) \quad ; \quad S_\nu(n) = \Psi_\nu(n) \Lambda_\nu(n) \Psi_\nu^T(n) = \sum_{j=1}^{M} \psi_{\nu,j}(n) \lambda_{\nu,j}(n) \psi_{\nu,j}^T(n)$$  \hspace{1cm} (12)

where $\tilde{\nu}_\nu(n)$: spectral-based turbulent principal coordinates as Fourier transforms of uncorrelated single-variate random processes; $M$: number of truncated turbulent modes ($M < N$); *,T denote to complex conjugate and transpose operation, respectively.

### 4 TURBULENT FIELD SIMULATION

Time series simulation of the spatially-correlated turbulent field at the structural discrete nodes is required for the time domain computation. The spectral-based simulations of i-th sub-process in the N-variate turbulent field can be expressed approximately using the POD-based spectral matrix decomposition and limited number of spectral eigenvectors 3)5):

$$\nu_i(t) \approx 2\sum_{j=1}^{M} \sum_{l=1}^{N} |\psi_{\nu,i}(n_l)| \sqrt{\lambda_{\nu,i}(n_l)} \Delta n_l \cos(2\pi n_l t + \theta_{\nu,i}(n_l) + \phi)$$  \hspace{1cm} (13)
where $l$: index of frequency point; $n$: frequency at moving point $l$; $N$: number of frequency intervals; $\Delta n$: frequency interval at point $l$; $\theta_{\nu}(n)$ : phase angle of complex eigenvector $\psi_{\nu}(n) = |\psi_{\nu}(n)| \exp(i \theta_{\nu}(n))$, determined as $\theta_{\nu}(n) = \tan^{-1}(\text{Im}(\psi_{\nu}(n))/\text{Re}(\psi_{\nu}(n)))$; $\phi$: phase angle considered as random variable uniformly distributed over $[0, 2\pi]$. In many cases, the eigenvectors are real and the frequency intervals are constant, it is simplified as follows:

$$v_i(t) \approx 2\sqrt{\Delta n} \sum_{j=1}^{N} \sum_{i=1}^{\hat{N}} \psi_{\nu}(n_i) \sqrt{\lambda_{\nu}(n_i)} \cos(2\pi n_i t + \phi)$$

where $i$: index of simulated sub-process; $\Delta n$: constantly frequency interval, $\Delta n = n_{up} / \hat{N}$, ($n_{up}$: upper cut-off frequency), $n_i = (i - 1)\Delta n$. Here, phase angles $\phi$ as random variables distributed uniformly over $[0, 2\pi]$ generated due to the Monte Carlo simulation technique.

5 **GUST RESPONSE PREDICTION**

5.1 **Time domain formulation using covariance proper transformation**

Multi-DOF motion equations of bridges subjected to the buffeting forces is expressed:

$$M\ddot{U} + C\dot{U} + KU = F_b(t)$$

where $M$, $C$, $K$: mass, damping and stiffness matrices, respectively; $U, \dot{U}, \ddot{U}$: deflection and its first-, second-order derivatives; $F_b(t)$: full-scale buffeting forces.

Decomposing into the modal space using the mass matrix-based normalized structural modal transformation as $U = \Phi \xi \approx \sum \phi \xi_i$, single-DOF motion equation in the $i$-th generalized structural coordinate due to the generalized buffeting forces can be obtained:

$$\ddot{\xi}_i + 2\zeta \omega_0 \dot{\xi}_i + \omega_0^2 \xi_i = \phi_i^T F_b(t)$$

where $\Phi$: modal matrix $\Phi = [\phi_1, \phi_2, ..., \phi_M]$; $\omega_0$, $\zeta$: circular frequency, damping ratio; $M$: number of truncated structural modes.

The full-scale buffeting forces in eq.(5) are used in which the two $N$-variate turbulent processes $u(t), w(t)$ are decomposed into uncorrelated turbulent principal coordinates using the covariance proper transformation as eq.(8) and eq.(9):

$$u(t) \approx \tilde{u}(t) = \Theta_u \tilde{\xi}_u(t) = \sum_{j=1}^{M} \theta_{uj} \tilde{\xi}_u(t); \quad w(t) \approx \tilde{w}(t) = \Theta_w \tilde{\xi}_w(t) = \sum_{j=1}^{M} \theta_{wj} \tilde{\xi}_w(t)$$

Thus, the single-DOF motion equation (16) is rewritten:

$$(\ddot{\xi}_i + 2\zeta \omega_0 \dot{\xi}_i + \omega_0^2 \xi_i) \approx \frac{1}{2} \rho U B \left[ \phi_i^T C_u \sum_{j=1}^{M} \theta_{uj} \tilde{\xi}_u(t) + \phi_i^T C_w \sum_{j=1}^{M} \theta_{wj} \tilde{\xi}_w(t) \right]$$

$$\ddot{\xi}_i + 2\zeta \omega_0 \dot{\xi}_i + \omega_0^2 \xi_i \approx \frac{1}{2} \rho U B \left[ \tilde{P}_u(t) \tilde{\xi}_u(t) + \tilde{P}_w(t) \tilde{\xi}_w(t) \right]$$

where $\tilde{P}_u(t) = \sum_{j=1}^{M} P_{uj}(t) = \sum_{j=1}^{M} \phi_i^T C_u \theta_{uj}(t)$; $\tilde{P}_w(t) = \sum_{j=1}^{M} P_{wj}(t) = \sum_{j=1}^{M} \phi_i^T C_w \theta_{wj}(t)$: cross modal coefficient matrices accounting for interrelation between the turbulent modes and the structural modes.

Time series and its derivatives of the generalized responses can be obtained by using direct integration methods such as the Newton-$\beta$ method ($\gamma = 1/2; \alpha = 1/4$). Finally, the global responses are determined.

5.2 **Frequency domain formulation using spectral proper transformation**
Power spectra of the generalized responses can be obtained thanks to the second-order Fourier transform with use of the spectral proper transformation as eq.(12):

\[
S_{ij}(n) = \frac{1}{2} \rho UB^2 \left[ H(n) \Phi \Psi_i^*(n) \Lambda_j(n) \Psi_j^T(n) K(n)^2 \Phi_i^T H(n)^T + H(n) \Phi C_{ij} \Psi_i(n) \Lambda_j(n) \Psi_j^T(n) K(n)^2 \Phi_i^T H(n)^T \right] \quad (19a)
\]

\[
S_{ij}(n) = \frac{1}{2} \rho UB^2 \left[ H(n) \hat{P}_{ii}(n) \Lambda_i(n) K(n)^2 \tilde{P}^R_i(n) H^T(n) + H(n) \hat{P}_{ij}(n) \Lambda_j(n) K(n)^2 \tilde{P}^R_i(n) H^T(n) \right] \quad (19b)
\]

where \( \hat{P}_{ij}(n) = \sum_{j=1}^{M} \hat{P}_{ij}(n) \) and \( \hat{P}_{ij}(n) \) imply for interrelation between j-th turbulent mode on i-th structural one; \( H(n) \) : frequency response function (FRF) matrix \( H(n) = diag(|H_i(n)|, |H_j(n)|, \ldots |H_M(n)|) \) in which term of \( |H_i(n)| \) denotes to FRF at natural frequency \( n_i \); \( K(n) \) : squared aerodynamic admittance function.

Spectra and mean square (RMS) of the global responses are obtained respectively. Finally, the root mean square of global responses with respect to vertical, longitudinal and rotational directions can be combined from single-modal responses due to the SRSS principle

\[
S_r(n) = \Phi S_{ij}(n) \Phi^T; \quad \sigma_r^2 = \int_0^\infty S_r(n) \, dn; \quad \sigma_r(n) = \sqrt{\sum_{i=1}^{3} \sigma_{r,i}^2}; \quad r = h, p, a
\]

where \( r \) denotes to displacement components: vertical (h), longitudinal (p), rotational (a); \( M \) : number of component modes in response combination;

6 NUMERICAL EXAMPLE AND DISCUSSIONS

A cable-stayed bridge was taken for demonstrating and investigating the above-mentioned computational procedures. 3D frame model was built using FEM, totally 30 discrete nodes on bridge deck (nodes 8, 23 at pylons). First ten modes were analyzed and given in Figure 3. It assumes that the buffeting forces only act on bridge deck. Aerodynamic static coefficients at bridge deck (nodes 8, 23 at pylons). First ten modes were analyzed and given in Figure 3. It

![Figure 3. Normalized structural modes: vertical (left) and rotational (right)](image)
At first, the time-domain response analysis of bridges has been carried out. Figure 4 shows the simulated time series of two turbulent processes u(t), w(t) of node 5 (mid of side span) and node 15 (mid of main span) in the totally time interval of 100 seconds at mean velocities U=20m/s and U=30m/s, respectively. Figure 5 shows the power auto spectra (PSD) of simulated time series of turbulent processes u(t), w(t) in some deck nodes and targeted auto spectra. There are good agreement between simulated auto spectra and targeted ones, and simulating procedure presented here has been validated for further studies. Time histories of turbulent-induced lift, moment and those of vertical displacement, rotational displacement in nodes 5 and 15 at also mean velocity U=20m/s during 100 seconds can be seen in Figure 6.

Figure 4. Simulated time series in nodes 5 & 15 at mean velocities U=20m/s (left) & U=30m/s (right)

Figure 5. Auto spectra of simulated time series in some deck nodes and targeted spectra at U=20m/s

Figure 6. Time histories of forces and global response in nodes 5 & 15 at U=20m/s
In the next step, response analysis of bridge in the frequency domain has been implemented using the POD-based spectral proper transformation. Figure 7 and Figure 8 show the first five spectral eigenvalues and first three spectral turbulent modes on 0.01÷10Hz, respectively. It notes that first spectral eigenvalue \( \lambda_1(n) \) exhibits much higher than the others on low frequency band 0.01÷0.2Hz of u-turbulence, 0.01÷0.5Hz of w-turbulence (see Figure 7). Spectral turbulent modes of u-,w-turbulences look like symmetrically and asymmetrically sinusoidal waves, moreover, shapes of the turbulent modes are unchanged among the natural frequencies of 0.61÷1.85Hz (see Figure 8).

Figure 7. First five spectral eigenvalues of spectral matrices \( S_u(n) \) (left) and \( S_w(n) \) (right)

Figure 8. First three spectral turbulent modes of \( S_u(n) \) (upper) and of \( S_w(n) \) (lower)

Figure 9. Spectra of global responses at node 15 vs. number of turbulent modes at U=20m/s
Figure 9 shows the power spectra of global responses at node 15 and effect of number of truncated turbulent modes. It is observed that no much different among cases of truncated turbulent modes, the first turbulent mode significantly contributes on spectra of responses. Figure 10 shows the root mean square of vertical and rotational displacements on whole bridge deck in comparison between time domain and frequency domain analyses as well as effect of number of turbulent modes. As seen that the first turbulent mode and the lowest modes in the frequency domain formulation contributes significantly on the global responses and play more important role than that in the time domain one.

7 CONCLUSION

New approach on the gust response prediction of bridges based on the covariance and spectral proper transformations has been discussed here. Fully-correlated turbulent field has been represented due to orthogonally covariance-based and spectral-based turbulent modes in which only limited number of lowest turbulent modes dominantly contributes on structural gust response. First turbulent mode seems enough to predict the gust response in the frequency domain, but more modes are required in the time domain. Further works focus on POD-based gust response prediction of long-span bridges with coupling of aeroelastic forces.

8 REFERENCES