Analysis of Algorithms

Data Structures and Algorithms

Acknowledgement:
These slides are adapted from slides provided with *Data Structures and Algorithms in C++*
Goodrich, Tamassia and Mount (Wiley, 2004)

Motivation

- What to do with algorithms?
  - Programmer needs to develop a working solution
  - Client wants problem solved efficiently
  - Theoretician wants to understand
- Why analyze algorithms?
  - To compare different algorithms for the same task
  - To predict performance in a new environment
  - To set values of algorithm parameters

Outline and Reading

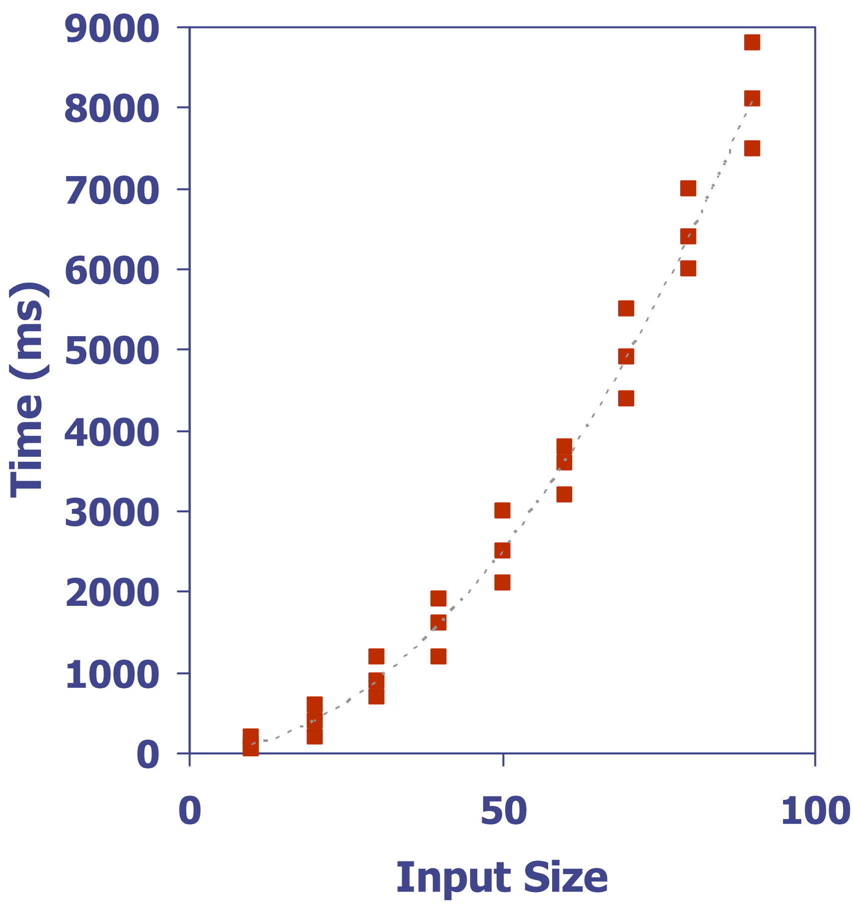
- Running time (§4.2)
- Pseudo-code
- Counting primitive operations (§4.2.2)
- Asymptotic notation (§4.2.3)
- Asymptotic analysis (§4.2.4)
- Case study (§4.2.5)

Running Time

- We are interested in the design of "good" data structures and algorithms.
- Measure of "goodness":
  - Running time (most important)
  - Space usage
- The running time of an algorithm typically grows with the input size, and is affected by other factors:
  - Hardware environments: processor, memory, disk.
  - Software environments: OS, compiler.
- Focus: **input size vs. running time.**

Experimental Studies

- Write a program implementing the algorithm
- Run the program with inputs of varying size and composition
- Use a method like System.currentTimeMillis() or clock() to get an accurate measure of the actual running time
- Plot the results



Measure Actual Running Time

```
//generate input data

//begin timing
clock_t k=clock();
clock_t start;
do                  //begin at new tick
   start = clock();
while (start == k);

//Run the test _num_itr times
for(int i=0; i<_num_itr; ++i) {
   //run the test once
}

//end timing
clock_t end = clock();

//calculate elapsed time
double elapsed_time = double(end - start) / double(CLOCKS_PER_SEC);
```

Limitations of Experiments

- It is necessary to implement the algorithm, which may be difficult and time consuming
- Results may not be indicative of the running time on other inputs not included in the experiment
- In order to compare two algorithms, the same hardware and software environments must be used

Theoretical Analysis

- Uses a high-level description of the algorithm instead of an implementation
- Takes into account all possible inputs
- Allows us to evaluate the speed of an algorithm independent of the hardware/software environment
- Goal: characterizes running time as a function of the input size n

Pseudocode

- High-level description of an algorithm
- More structured than English prose
- Less detailed than a program source code
- Preferred notation for describing algorithms
- Hides program design issues

Example: find max element of an array

Algorithm *arrayMax*(A, n)
  Input array A of n integers
  Output maximum element of A

  $currentMax \leftarrow A[0]$
  for $i \leftarrow 1$ **to** $n - 1$ **do**
    if $A[i] > currentMax$ **then**
      $currentMax \leftarrow A[i]$
  return $currentMax$

Pseudocode Details

- Control flow
  - **if** … **then** … [**else** …]
  - **while** … **do** …
  - **repeat** … **until** …
  - **for** … **do** …
  - Indentation replaces braces
- Method declaration

  Algorithm *method* (*arg* [, *arg*…])

  Input …

  Output …
- Method call

  var.method (*arg* [, *arg*…])
- Return value

  return *expression*
- Expressions

  $\leftarrow$ Assignment (like = in C++/Java)

  = Equality testing (like == in C++/Java)

  n^2 Superscripts and other mathematical formatting allowed

Primitive Operations

- Basic computations performed by an algorithm
- Identifiable in pseudocode
- Largely independent from the programming language
- Exact definition not important
- Assumed to take a constant execution time

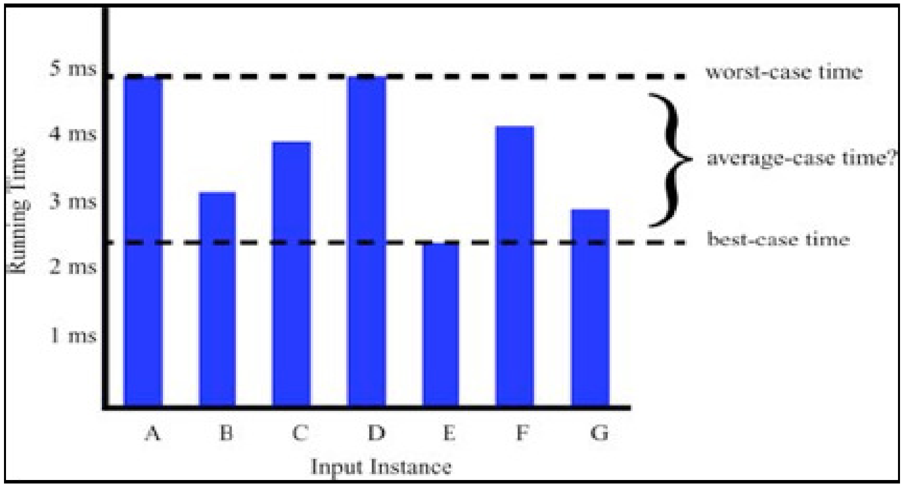
- Examples:
  - Performing an arithmetic operation
  - Comparing two numbers
  - Assigning a value to a variable
  - Indexing into an array
  - Calling a method
  - Returning from a method

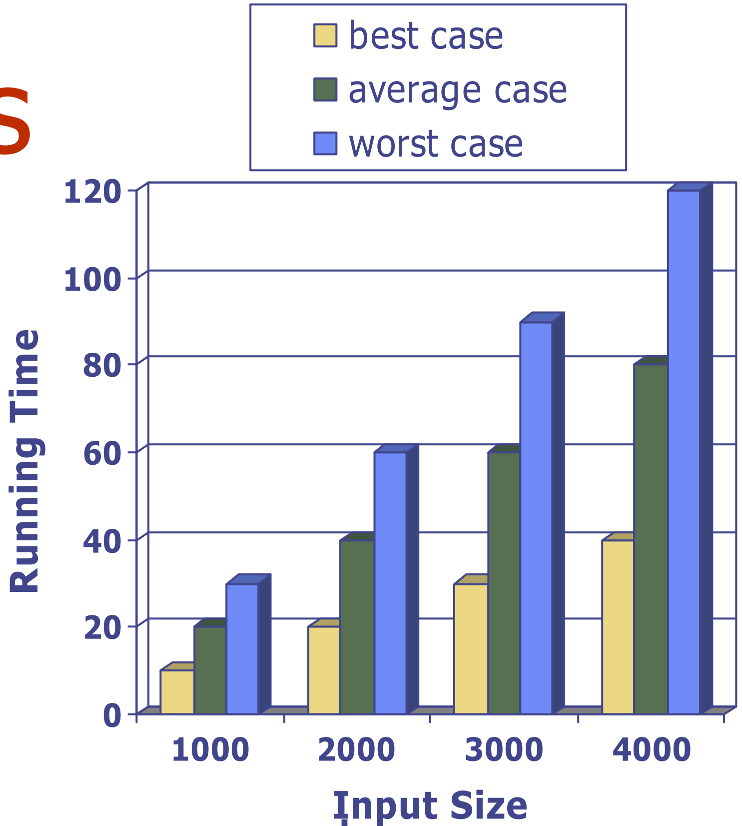
Counting Primitive Operations

- By inspecting the pseudocode, we can determine the maximum number of primitive operations executed by an algorithm, as a function of the input size

Algorithm *arrayMax*(A, n)	# operations
$currentMax \leftarrow A[0]$	2
for $i \leftarrow 1$ **to** $n - 1$ **do**	$2n$
if $A[i] > currentMax$ **then**	$2(n - 1)$
$currentMax \leftarrow A[i]$	$2(n - 1)$
{ increment counter i }	$2(n - 1)$
return $currentMax$	1
Total	$8n - 2$

Worst case analysis





- Average case analysis is difficult for many problems:
  - Probability distribution of inputs.
- We focus on the worst case analysis
  - Easier
  - If an algorithm does well in the worst-case, it will perform well on all cases

Estimating Running Time

- Algorithm *arrayMax* executes $8n - 2$ primitive operations in the worst case. Define:

  a = Time taken by the fastest primitive operation

  b = Time taken by the slowest primitive operation
- Let $T(n)$ be worst-case time of *arrayMax*. Then

$$a\,(8n - 2) \le T(n) \le b(8n - 2)$$

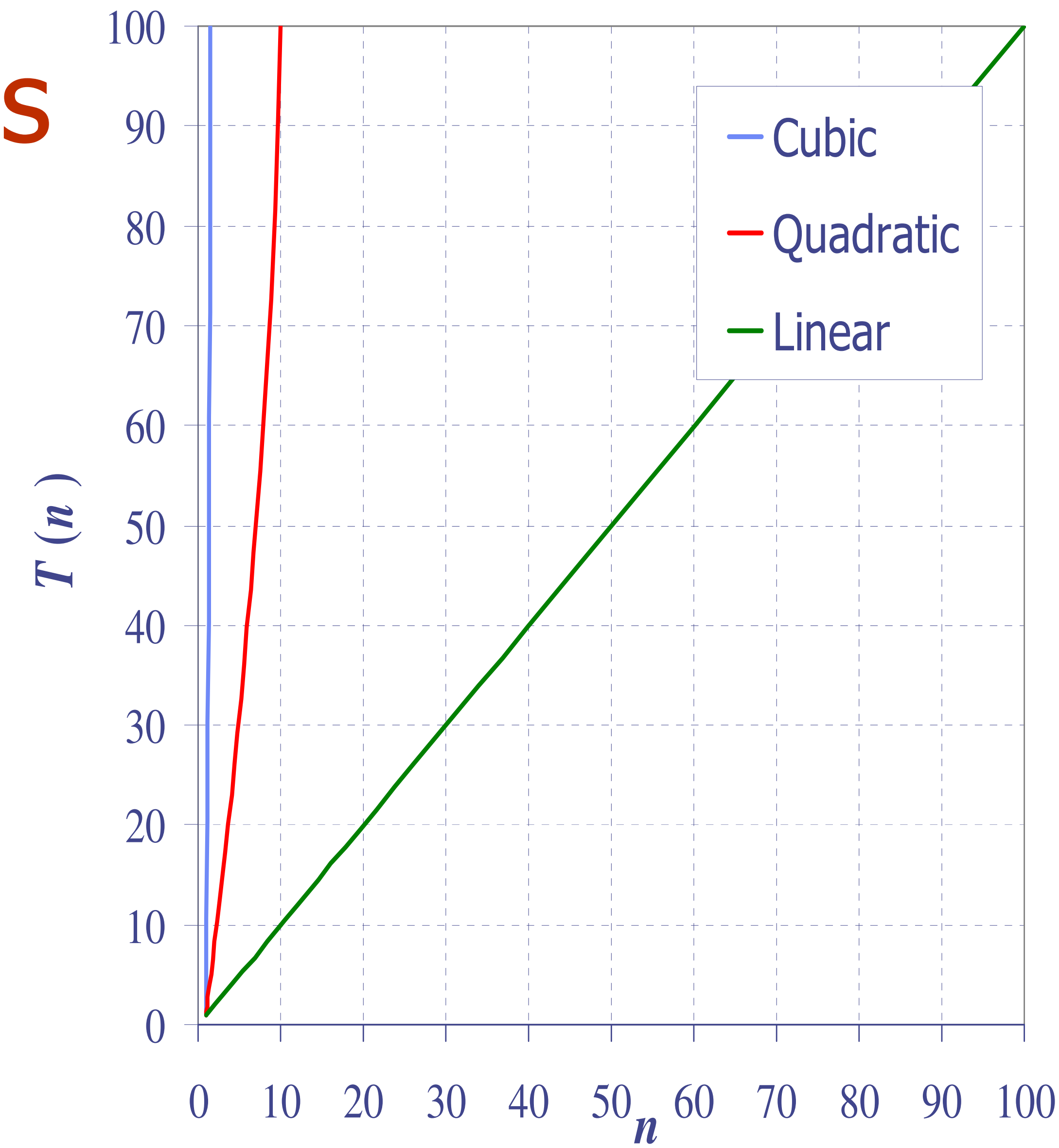
- Hence, the running time $T(n)$ is bounded by two linear functions

Growth Rate of Running Time

- Changing the hardware/ software environment
  - affects $T(n)$ by a constant factor, but
  - does not alter the growth rate of $T(n)$
- The linear growth rate of the running time $T(n)$ is an intrinsic property of algorithm $\boldsymbol{arrayMax}$.

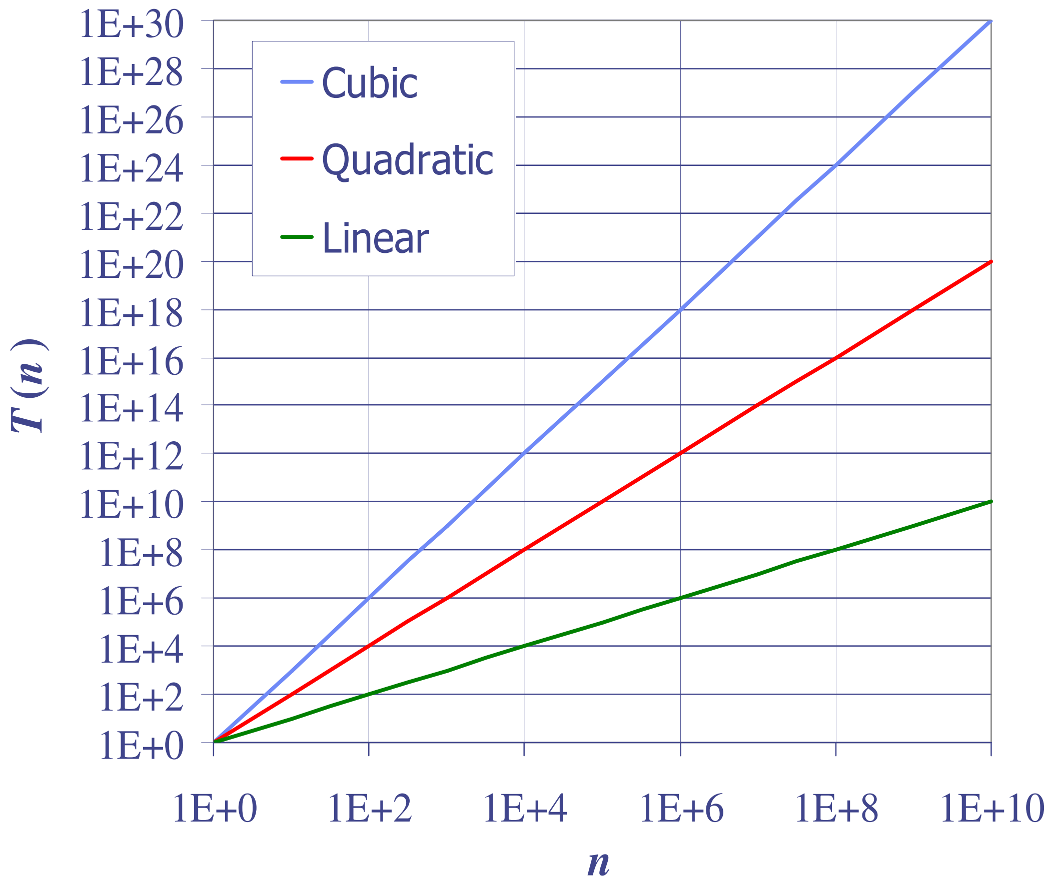
Growth Rates

- Growth rates of functions:
  - Linear ≈ n
  - Quadratic ≈ n^2
  - Cubic ≈ n^3



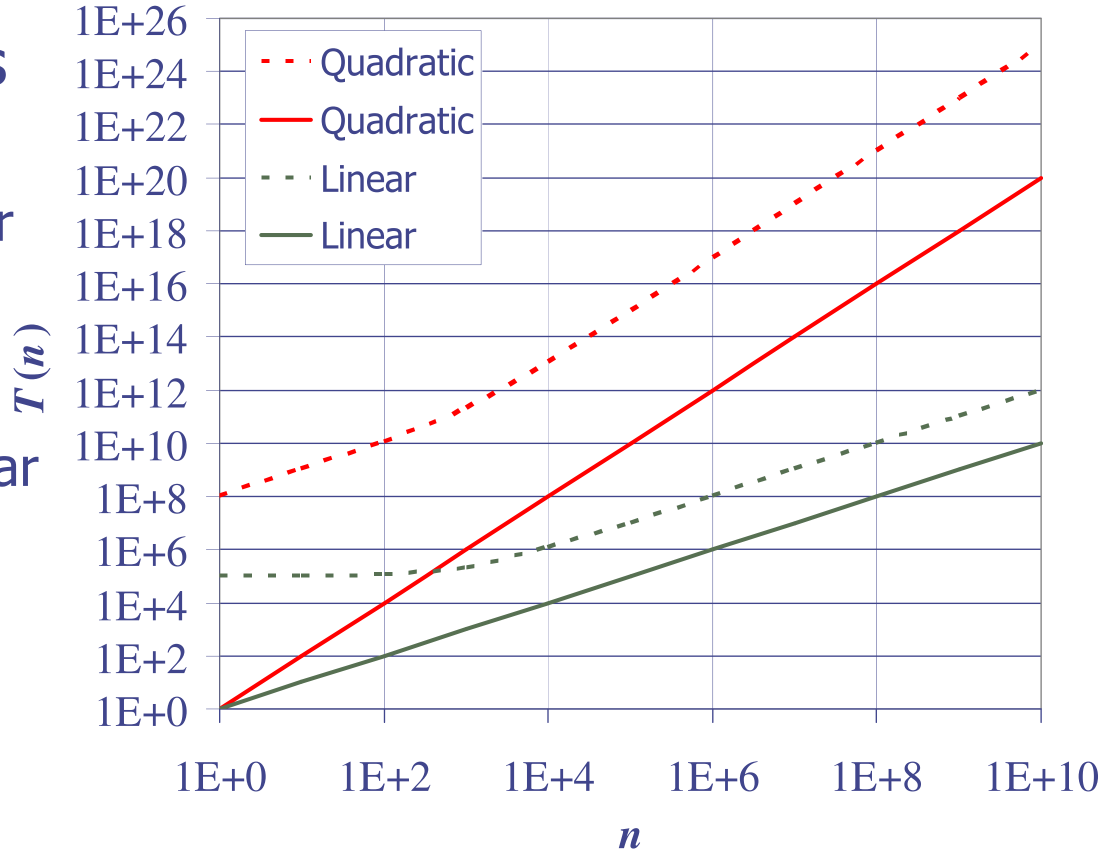
Growth Rates

- Growth rates of functions:
  - Linear ≈ n
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
- In a log-log chart, the slope of the line corresponds to the growth rate of the function



Constant Factors

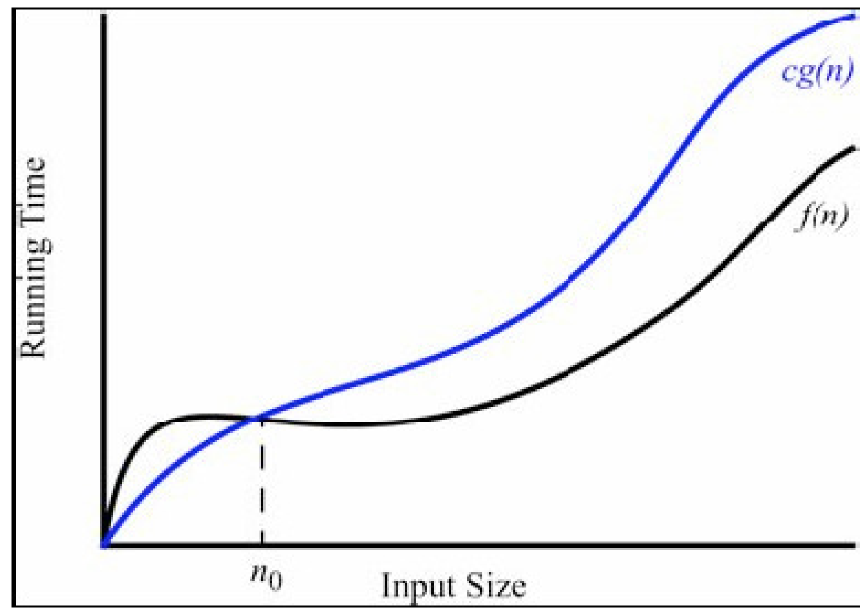
- The growth rate is not affected by
  - constant factors or
  - lower-order terms
- Examples
  - $10^2 n + 10^5$ is a linear function
  - $10^5 n^2 + 10^8 n$ is a quadratic function



Big-Oh Notation Example

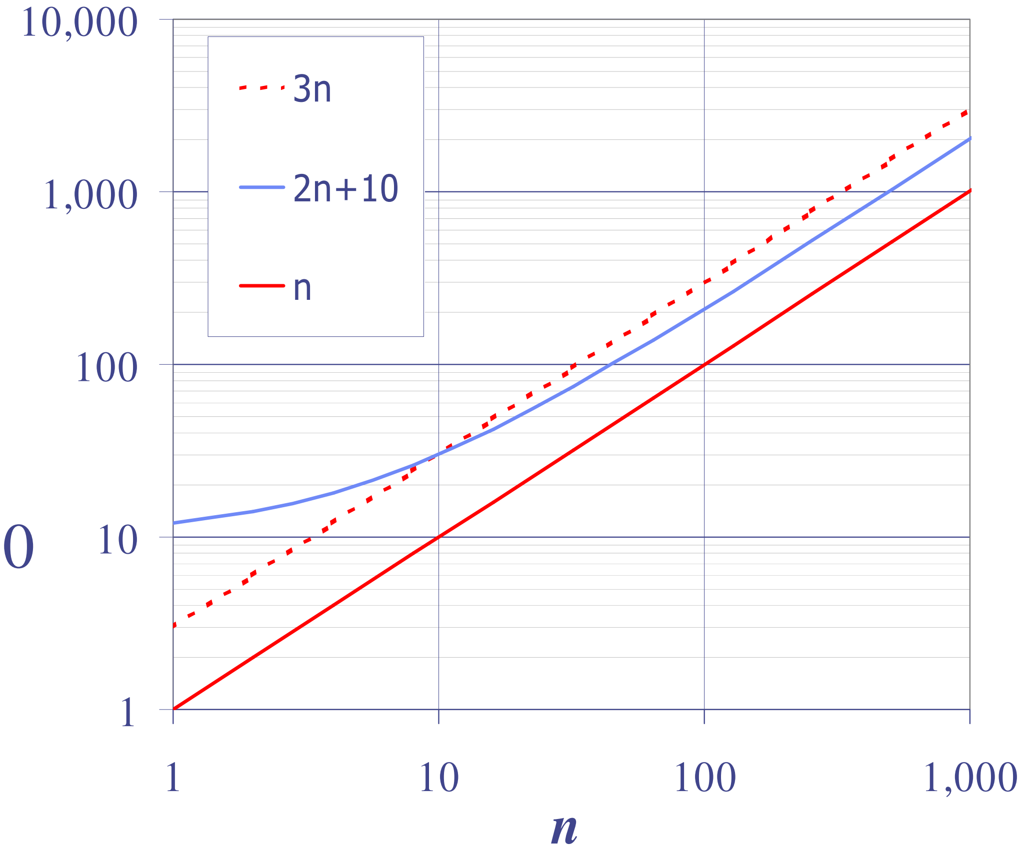
- Given functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there are positive constants c and n_0 such that

  $f(n) \leq cg(n)$ for $n \geq n_0$



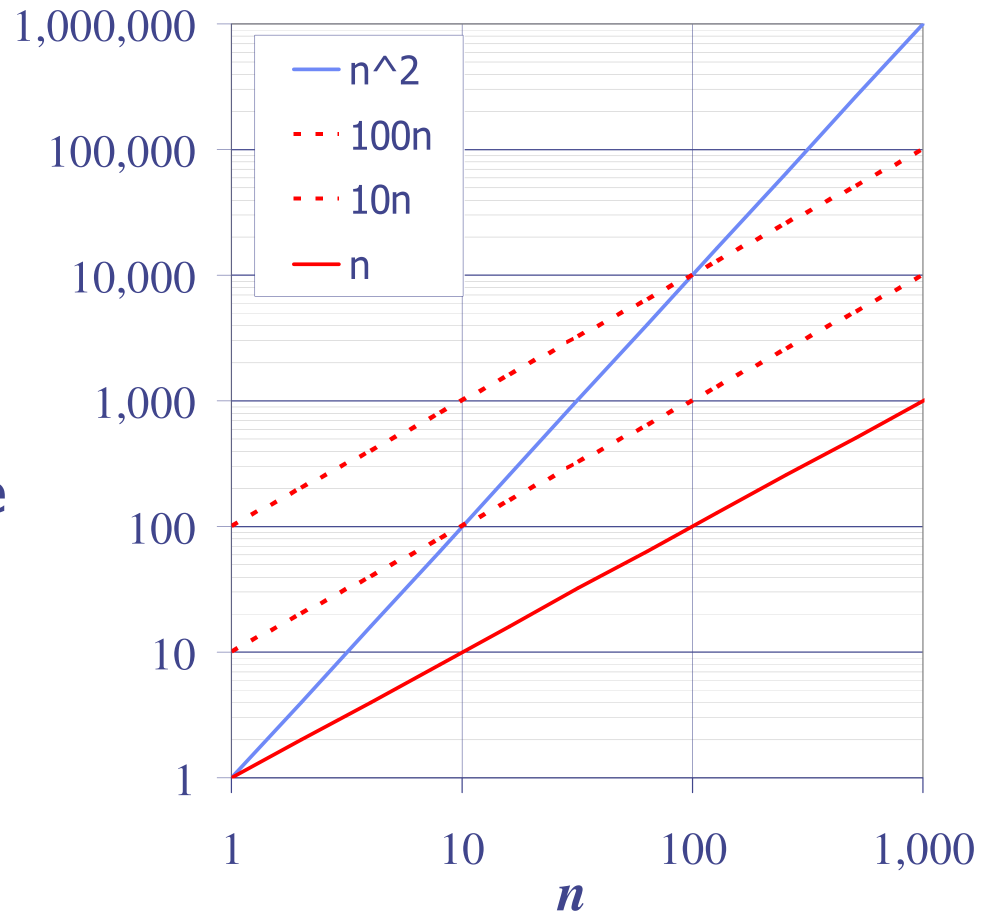
Big-Oh Notation Example

- Example:
  $2n + 10$ is $O(n)$
  - $2n + 10 \leq cn$
  - $(c - 2)\, n \geq 10$
  - $n \geq 10/(c - 2)$
  - Pick $c = 3$ and $n_0 = 10$



Big-Oh Notation Example (cont.)

- Example: the function n^2 is not $O(n)$
  - $n^2 \le cn$
  - $n \le c$
  - The above inequality cannot be satisfied since c must be a constant



More Big-Oh Examples

- 7n-2

  7n-2 is O(n)

  need $c > 0$ and $n_0 \geq 1$ such that $7n-2 \leq c \cdot n$ for $n \geq n_0$

  this is true for $c = 7$ and $n_0 = 1$

- $3n^3 + 20n^2 + 5$

  $3n^3 + 20n^2 + 5$ is $O(n^3)$

  need $c > 0$ and $n_0 \geq 1$ such that $3n^3 + 20n^2 + 5 \leq c \cdot n^3$ for $n \geq n_0$

  this is true for $c = 4$ and $n_0 = 21$

- 3 log n + 5

  3 log n + 5 is O(log n)

  need $c > 0$ and $n_0 \geq 1$ such that $3 \log n + 5 \leq c \cdot \log n$ for $n \geq n_0$

  this is true for $c = 8$ and $n_0 = 2$

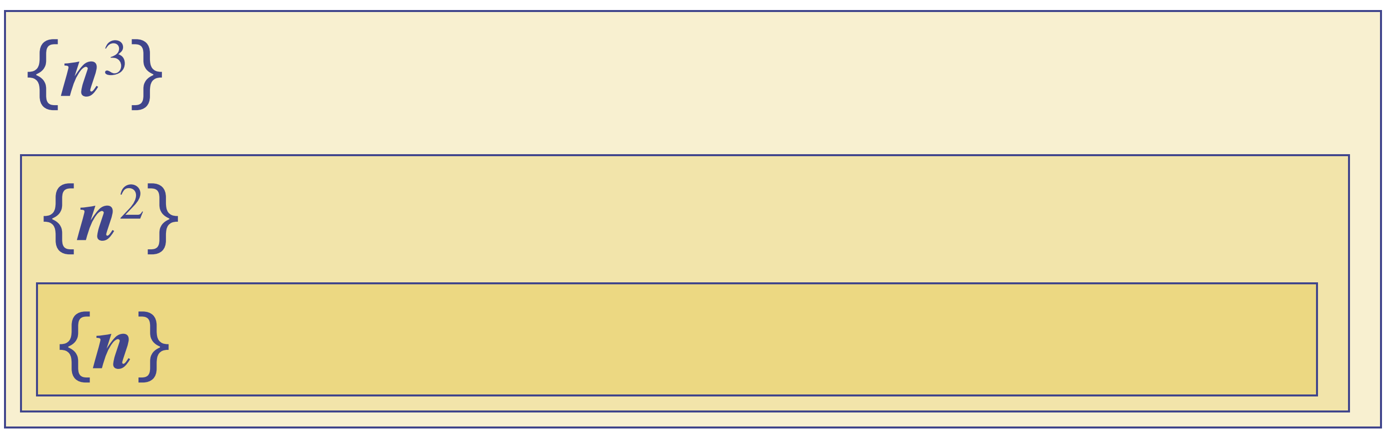
Big-Oh and Growth Rate

- The big-Oh notation gives an upper bound on the growth rate of a function
- The statement "$f(n)$ is $O(g(n))$" means that the growth rate of $f(n)$ is no more than the growth rate of $g(n)$
- We can use the big-Oh notation to rank functions according to their growth rate

	$f(n)$ is $O(g(n))$	$g(n)$ is $O(f(n))$
$g(n)$ grows more	Yes	No
$f(n)$ grows more	No	Yes
Same growth	Yes	Yes

Classes of Functions

- Let $\{g(n)\}$ denote the class (set) of functions that are $O(g(n))$
- We have
  $\{n\} \subset \{n^2\} \subset \{n^3\} \subset \{n^4\} \subset \{n^5\} \subset \ldots$
  where the containment is strict



Big-Oh Rules

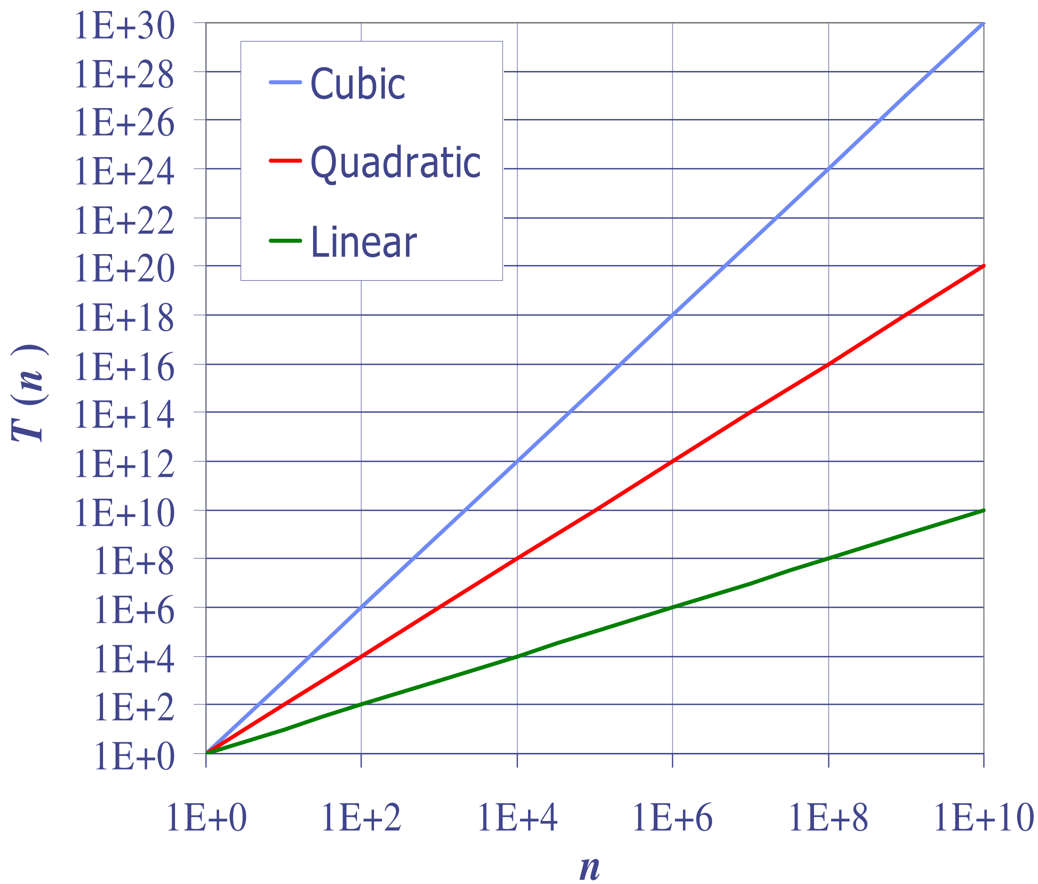
- If is $f(n)$ a polynomial of degree d, then $f(n)$ is $O(n^d)$, i.e.,
  1. Drop lower-order terms
  2. Drop constant factors
- Use the smallest possible class of functions
  - Say "$2n$ is $O(n)$" instead of "$2n$ is $O(n^2)$"
- Use the simplest expression of the class
  - Say "$3n + 5$ is $O(n)$" instead of "$3n + 5$ is $O(3n)$"

Asymptotic Algorithm Analysis

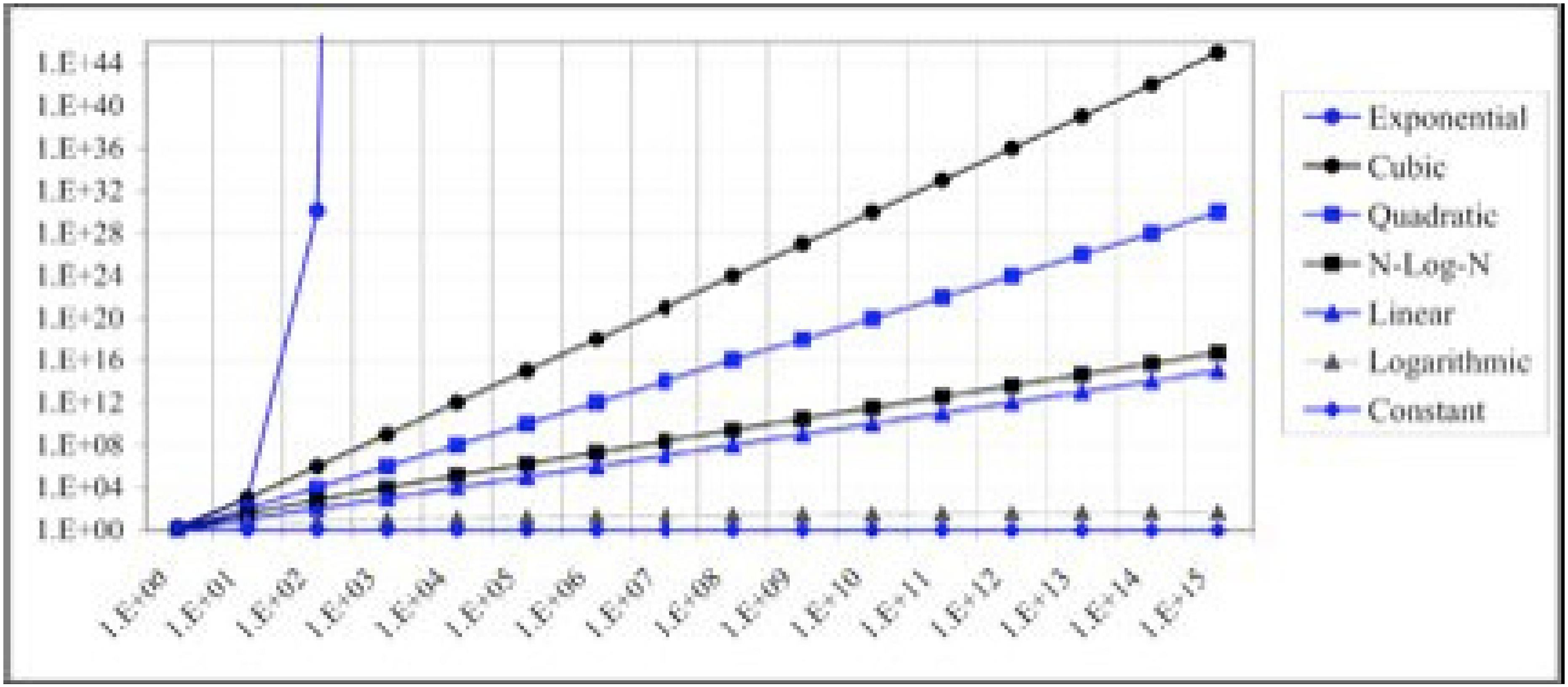
- The asymptotic analysis of an algorithm determines the running time in big-Oh notation
- To perform the asymptotic analysis
  - We find the worst-case number of primitive operations executed as a function of the input size
  - We express this function with big-Oh notation
- Example:
  - We determine that algorithm ***arrayMax*** executes at most $8n - 2$ primitive operations
  - We say that algorithm ***arrayMax*** "runs in $O(n)$ time"
- Since constant factors and lower-order terms are eventually dropped anyhow, we can disregard them when counting primitive operations

Seven Important Functions

- Seven functions that often appear in algorithm analysis:
  - Constant ≈ *1*
  - Logarithmic ≈ log n
  - Linear ≈ n
  - N-Log-N ≈ n log n
  - Quadratic ≈ n^2
  - Cubic ≈ n^3
  - Exponential ≈ 2^n



Seven Important Functions



Asymptotic Analysis

Running Time	**Maximum Problem Size (n)**		
	1 second	1 minute	1 hour
400n	2,500	150,000	9,000,000
20nlogn	4,096	166,666	7,826,087
$2n^2$	707	5,477	42,426
n^4	31	88	244
2^n	19	25	31

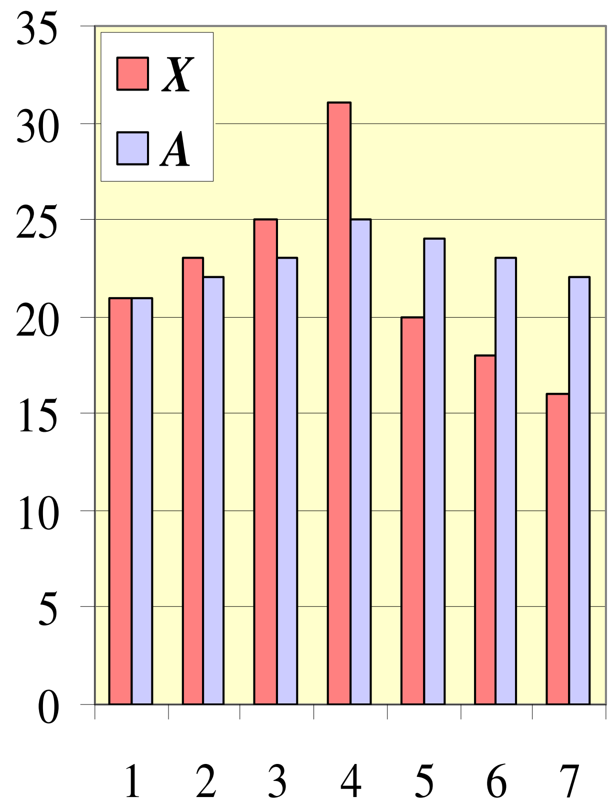
- Caution: $10^{100}n$ vs. n^2

Computing Prefix Averages

- We illustrate asymptotic analysis with two algorithms for prefix averages
- The i-th prefix average of an array X is average of the first $(i+1)$ elements of X

$$A[i] = (X[0] + X[1] + \ldots + X[i])/(i+1)$$

- Problem: compute the array A of prefix averages of another array X
- Applications in economics and statistics



Prefix Averages (Quadratic)

- The following algorithm computes prefix averages in quadratic time by applying the definition

Algorithm *prefixAverages1*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow X[0]$	n
for $j \leftarrow 1$ **to** i **do**	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + X[j]$	$1 + 2 + \ldots + (n - 1)$
$A[i] \leftarrow s / (i + 1)$	n
return A	1

Arithmetic Progression

- The running time of ***prefixAverages1*** is $O(1 + 2 + \ldots + n)$

  or

  $O(\, n(n + 1) / 2 \,)$
- Thus, the algorithm ***prefixAverages1*** runs in $O(n^2)$ time

Prefix Averages (Linear)

- The following algorithm computes prefix averages in linear time by keeping a running sum

Algorithm *prefixAverages2*(X, n)	
Input array X of n integers	
Output array A of prefix averages of X	#operations
$A \leftarrow$ new array of n integers	n
$s \leftarrow 0$	1
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow s + X[i]$	n
$A[i] \leftarrow s / (i + 1)$	n
return A	1

- Algorithm ***prefixAverages2*** runs in $O(n)$ time

Relatives of Big-Oh, Intuition for Asymptotic Notation

Big-Oh

- f(n) is O(g(n)) if f(n) is asymptotically **less than or equal** to g(n)

Big-Omega

- f(n) is Ω(g(n)) if f(n) is asymptotically **greater than or equal** to g(n)
  - f(n) is Ω(g(n)) if there is a constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $f(n) \geq c \bullet g(n)$ for $n \geq n_0$

Big-Theta

- f(n) is Θ(g(n)) if f(n) is asymptotically **equal** to g(n)
  - f(n) is Θ(g(n)) if there are constants $c' > 0$ and $c'' > 0$ and an integer constant $n_0 \geq 1$ such that $c' \bullet g(n) \leq f(n) \leq c'' \bullet g(n)$ for $n \geq n_0$