Checking Hybrid Automata for Linear Duration Invariants *

Li Xuandong, Dang Van Hung, and Zheng Tao

International Institute for Software Technology
The United Nations University, P.O.Box 3058, Macau
E-Mail: {lxd, dvh, zt}@iist.unu.edu

Abstract. In this paper, we consider the problem of checking hybrid systems modelled by hybrid automata for a class of real-time properties represented by *linear duration invariants*, which are constructed from linear inequalities of integrated durations of system states. Based on linear programming, an algorithm is developed for solving the problem for a class of hybrid automata.
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1 Introduction

A hybrid system consists of a discrete component and a continuous component. Since the methods to analyse the discrete components are different from the methods to analyse the continuous components, and since the interface between the two components is complicated, it is very difficult to analyse hybrid systems. Therefore, very often, one has to restrict oneself to some smaller class of hybrid systems so that the problems of concern can be solved efficiently. One of the classes of hybrid systems that has received a great deal of attention in the literature is the class of linear hybrid systems [1,2].

A linear hybrid system can be modelled by a linear hybrid automaton [1]. Informally, a hybrid automaton is a conventional automaton extended with a set of variables, which are used to model the state of the continuous component of hybrid systems and are assumed to be piecewise linear functions of time. The states of the automaton called *locations* are assigned with a change rate for each variable, such as $\dot{x} = w$ (x is a variable, w is a real number), and the transitions of the automaton are labelled with constraints on the variables such as $a \leq x \leq b$ and /or with reset actions such as $x := c$ (x is a variable, a, b, and c are real numbers). The automaton starts at one of the initial locations with all variables initialised to their initial values. As time progresses, the values of all variables change continuously according to the rate associated with the current location. At any time, the system can change its current location from s to s' provided

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that there is a transition ρ from s to s' whose labelling conditions are satisfied by the current value of the variables. With a location change by a transition ρ, all the variables are reset to the new value accordingly by the reset actions labelled on ρ. Transitions are assumed to be instantaneous.

Let us consider an example of a water-level monitor in [1]. The water level in a tank is controlled through a monitor, which continuously senses the water level and turns a pump on and off. The water level changes as a piecewise-linear function of time. When the pump is off, the water level falls by two inches per second; when the pump is on, the water level rises by one inch per second. Suppose that initially the water level is one inch and the pump is on. The requirement for the monitor is that the water level should be kept in between one and 12 inches. There is a delay of two seconds from the time that the monitor signals to change the status of the pump to the time that the change becomes effective. Thus the monitor must signal to turn the pump on (off) at least two seconds before the water level falls to 1 inch (reaches 12 inches). The system is modelled by the hybrid automaton depicted in Figure 1. The automaton has four locations. In the locations s_1 and s_2, the pump is on; in the locations s_3 and s_4, the pump is off. The variable y is used to model the water-level, and x is used to specify the delays: whenever the control is in location s_2 or s_4, the value of x indicates how long the signal to switch the pump off or on has been sent.

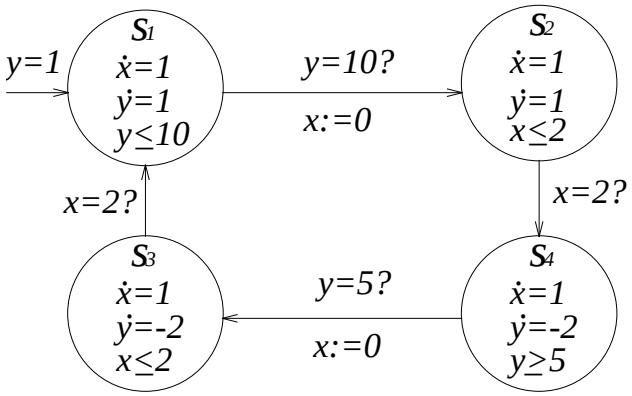

Fig. 1. A hybrid automaton modelling a water-level monitor

In this paper, we are concerned by the problem of checking automatically a linear hybrid automaton for a real time property, which is expressed by *linear duration invariants*. Linear duration invariants [4] are constructed from linear inequalities of integrated durations of system states. They form an important class of Duration Calculus (DC) [3] formulas. In DC, states are modelled as Boolean functions from reals (representing continuous time) to $\{0, 1\}$, where 1 denotes state presence, and 0 denotes state absence. For a state S, the interval

variable $\int S$ of DC is a function from bounded and closed intervals to reals which stands for the accumulated presence time (duration) of state S over the intervals, and is defined formally by $\int S[a,b] \hat{=} \int_a^b S(t)dt$, where $[a,b]$ $(b \geq a)$ is a bounded interval of time. A linear duration invariant $\mathcal{D}$ in DC is of the form

$$T \geq \int 1 \geq t \Rightarrow \bigwedge_{j=1}^{k} (\sum_{i=1}^{n} c_{ij} \int S_i \leq M_j)\,,$$

where T, t, c_{ij}, M_j are real numbers (T may be ∞).

The meaning of a linear duration invariant $\mathcal{D}$ is that: if the system is observed for an interval of time satisfying the premise of $\mathcal{D}$, then the duration of the system states must satisfy the consequence of $\mathcal{D}$. It turns out that many real-time properties can be written as a linear duration invariant.

For example, the requirement of the water-level monitor, which is that the monitor must keeps the water level in between 1 and 12 inches, can be expressed by linear duration invariants as well. We know that when the control is in locations s_1 or s_2, the water level rises 1 inch per second, and when the control is in locations s_3 or s_4, the water level falls by 2 inch per second. Furthermore, for an interval $[0,t]$, the accumulated time that the system stays in s_1 or s_2 is $\int s_1 + \int s_2$, and the accumulated time that the system stays in s_3 or s_4 is $\int s_3 + \int s_4$. Therefore, the water level at time t, given that at the beginning the water level is one inch, is $1 + \int s_1 + \int s_2 - 2(\int s_3 + \int s_4)$. Hence, the requirement for the water-level monitor can be described by the following linear duration invariants

$$0 \leq \int 1 \leq \infty \Rightarrow 1 + \int s_1 + \int s_2 - 2(\int s_3 + \int s_4) \leq 12\,;$$
$$0 \leq \int 1 \leq \infty \Rightarrow 1 + \int s_1 + \int s_2 - 2(\int s_3 + \int s_4) \geq 1\,.$$

Now, the problem we are concerned in this paper can be formulated as follows. Given a hybrid automaton A, given a linear duration invariant $\mathcal{D}$, decide efficiently whether A satisfy $\mathcal{D}$.

The problem has attracted a great deal of attention. In [6] the authors have solved the problem for a subclass of integration graph using mixed integer/linear programming techniques which inherit very high complexity. In [4], the authors have solved the problem for a simple class of real time automata using linear programming techniques, which is well established. Because of the advantages of the approach of [4] in comparison to the others, in [9] we have generalised it to a subclass of timed automata [5].

In this paper, by developing the techniques in [4,9] further, we show that by linear programming techniques the problem can be solved totally for a well-formed subclass of linear hybrid automata.

The paper is organised as follows. In the next section, we introduce the notion of hybrid regular expressions to express the behaviour of hybrid automata. Our model-checking algorithm is presented in Section 3. The last section is the conclusion of the paper.

2 Hybrid Regular Expressions

A traditional way to express the behaviour of an automaton is to use regular expressions. In this section, we extend the traditional regular expressions with time constraints and use them as a language to describe the behaviour of linear hybrid systems. The extended notation will be called *Hybrid Regular Expression* (HRE). While a regular expression over a set of states (alphabet) is a finite representation of a (infinite) set of sequences of states, an HRE will be a finite representation of a set of timed sequences of states.

Let V be a finite set, R^+ be the set of nonnegative real numbers. Each element of V is called a location. A finite sequence $(s_1,t_1)\hat{}(s_2,t_2)\hat{}\ldots\hat{}(s_m,t_m)$ of elements in $V\times R^+$ is called a timed sequence over V. In this paper, we use $\hat{}$ to denote the concatenation of the sequences. The occurrence time $\tau(\sigma)$ of a timed sequence $\sigma=(s_1,t_1)\hat{}(s_2,t_2)\hat{}\ldots\hat{}(s_m,t_m)$ over V is defined by $\tau(\sigma)=\sum_{i=1}^{m}t_i$.

A timed sequence $(s_1,t_1)\hat{}(s_2,t_2)\hat{}\ldots\hat{}(s_m,t_m)$ represents a behaviour of a system that the system starts at the state s_1, stays there for t_1 time units, then changes to s_2 and stays in s_2 for t_2 time units, and so on. The values $t_1,t_2,\ldots$ have to satisfy some time constraints enforced by the system. These time constraints must be incorporate into the finite representation of the system behaviours. By incorporating time constraints into regular expressions, we get hybrid regular expressions.

An HRE $\mathcal{R}$ and the language $\mathcal{L}(\mathcal{R})$ represented by $\mathcal{R}$ over a finite set V of states are defined recursively as follows.

Definition 1.

1. *ε is an HRE, and $\mathcal{L}(\varepsilon)=\{\varepsilon\}$.*
2. *Let $v_1,v_2,\ldots,v_m\in V$ $(m\geq 1)$, and Δ be a set of linear inequalities on $\lambda_1,\lambda_2,\ldots,\lambda_m$ of the form $a\leq c_1\lambda_1+c_2\lambda_2+\ldots+c_m\lambda_m\leq b$, where a,b, and c_i $(1\leq i\leq m)$ are real numbers. Then $(v_1\hat{}v_2\hat{}\ldots\hat{}v_m,\Delta)$ is an HRE, and*

$$\mathcal{L}((v_1\hat{}v_2\hat{}\ldots\hat{}v_m,\Delta))=$$
$$\left\{(v_1,t_1)\hat{}(v_2,t_2)\hat{}\ldots\hat{}(v_m,t_m)\;\middle|\;\begin{array}{c}t_1,\ldots,t_m\geq 0 \text{ such that for all}\\ a\leq\sum_{i=1}^{m}c_i\lambda_i\leq b\in\Delta,\ a\leq\sum_{i=1}^{m}c_it_i\leq b\end{array}\right\}.$$

   When $\Delta=\phi$, $(v_1\hat{}v_2\hat{}\ldots\hat{}v_m,\Delta)$ is taken to be $v_1\hat{}v_2\hat{}\ldots\hat{}v_m$.
3. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are HREs, then $\mathcal{R}_1\hat{}\mathcal{R}_2$ is an HRE, and*

$$\mathcal{L}(\mathcal{R}_1\hat{}\mathcal{R}_2)=\{\sigma_1\hat{}\sigma_2\mid\sigma_1\in\mathcal{L}(\mathcal{R}_1),\ \sigma_2\in\mathcal{L}(\mathcal{R}_2)\}.$$

4. *If $\mathcal{R}_1$ and $\mathcal{R}_2$ are HREs, then $\mathcal{R}_1\oplus\mathcal{R}_2$ is an HRE, and*

$$\mathcal{L}(\mathcal{R}_1\oplus\mathcal{R}_2)=\mathcal{L}(\mathcal{R}_1)\cup\mathcal{L}(\mathcal{R}_2).$$

5. *If $\mathcal{R}$ is an HRE, then $\mathcal{R}^*$ is an HRE, and*

$$\mathcal{L}(\mathcal{R}^*)=\{\sigma_1\hat{}\ldots\hat{}\sigma_m\mid m\geq 0 \text{ and } \bigwedge_{i=1}^{m}(\sigma_i\in\mathcal{L}(\mathcal{R}))\},$$

   where $\sigma_1\hat{}\ldots\hat{}\sigma_m\hat{=}\epsilon$ when $m=0$.

6. *If $\mathcal{R}$ is an HRE, $a \in R^+, b \in R^+ \cup \{\infty\}, a \leq b$, and $b > 0$, then $(\mathcal{R}, [a, b])$ is an HRE (when $b = \infty$, $(\mathcal{R}, [a, b])$ is taken to be $(\mathcal{R}, [a, \infty))$), and*

$$\mathcal{L}((\mathcal{R}, [a, b])) = \{\sigma \mid \sigma \in \mathcal{L}(\mathcal{R}) \text{ and } a \leq \tau(\sigma) \leq b\}.$$

□

Although the traditional regular expressions are powerful enough to describe the behaviour of finite automata, it is not the case for HREs to describe the behaviour of all linear hybrid automata. The reason is that the constraints for continuous variables (occurring in Δ) can be put for a fixed finite sequence of states only. Nevertheless, it is simple and powerful enough to express the real-time behaviour of many hybrid systems encountered in practice.

For example, the behaviour of the linear hybrid automaton (Fig. 1) modelling the water level monitor in the introduction can be represented by the following HRE $\mathcal{R}_w$:

$$\begin{aligned}\mathcal{R}_w = \varepsilon \oplus (s_1, [0, 9]) \oplus (s_1, [9, 9])\hat{}(s_2, [0, 2]) \oplus \mathcal{R}_1 \\ \oplus \mathcal{R}_2\hat{}\mathcal{R}_3^*\hat{}((s_4, [0, 2]) \oplus \mathcal{R}_4 \oplus \mathcal{R}_5 \oplus \mathcal{R}_6)\end{aligned}$$

where

$$\begin{aligned}
\mathcal{R}_1 &= (s_1\hat{}s_2\hat{}s_3, \{\lambda_1 = 9,\ \lambda_2 = 2,\ 2\lambda_3 - \lambda_2 \leq 5\}) \\
\mathcal{R}_2 &= (s_1\hat{}s_2\hat{}s_3, \{\lambda_1 = 9,\ \lambda_2 = 2,\ 2\lambda_3 - \lambda_2 = 5\}) \\
\mathcal{R}_3 &= (s_4\hat{}s_1\hat{}s_2\hat{}s_3, \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 = 5,\ \lambda_3 = 2,\ 2\lambda_4 - \lambda_3 = 5\}) \\
\mathcal{R}_4 &= (s_4\hat{}s_1, \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 \leq 5\}) \\
\mathcal{R}_5 &= (s_4\hat{}s_1\hat{}s_2, \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 = 5,\ 0 \leq \lambda_3 \leq 2\}) \\
\mathcal{R}_6 &= (s_4\hat{}s_1\hat{}s_2\hat{}s_3, \\
&\qquad \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 = 5,\ \lambda_3 = 2,\ -5 \leq \lambda_3 - 2\lambda_4\}).
\end{aligned}$$

Since HREs form a very simple formalism to model hybrid systems, hopefully many problems are decidable for the class of hybrid systems defined by HREs. In next section, we show that checking an HRE for a linear duration invariant is decidable for the class, and we will give an efficient algorithm for solving the problem. In the rest of this section, we give some concepts concerning HREs that will be used for presenting our algorithms.

Definition 2. *For an HRE $\mathcal{R}$, the sub-expressions of $\mathcal{R}$ are defined recursively by:*

1. *$\mathcal{R}$ is a sub-expression of $\mathcal{R}$.*
2. *If $\mathcal{R} = \mathcal{R}_1\hat{}\mathcal{R}_2$ or $\mathcal{R} = \mathcal{R}_1 \oplus \mathcal{R}_2$, where $\mathcal{R}_1$ and $\mathcal{R}_2$ are HREs, then all the sub-expressions of $\mathcal{R}_1$ and $\mathcal{R}_2$ are sub-expressions of $\mathcal{R}$.*
3. *If $\mathcal{R} = \mathcal{R}_1^*$ or $\mathcal{R} = (\mathcal{R}_1, [a, b])$, where $\mathcal{R}_1$ is an HRE, then all the sub-expressions of $\mathcal{R}_1$ are sub-expressions of $\mathcal{R}$.* □

For an HRE $\mathcal{R}$, if $\mathcal{L}(\mathcal{R}) = \phi$, then $\mathcal{R}$ is said to be *empty*. For example,

$$((e_1\hat{}e_2, \{3 \leq \lambda_1 \leq 4, 4.5 \leq \lambda_2 \leq 5\}), [4, 7])$$

is an empty HRE. For an empty HRE $\mathcal{R}_1$ and for an HRE $\mathcal{R}$, it follows from the definition of HREs that

$$\begin{array}{l}\mathcal{L}(\mathcal{R}_1\hat{}\mathcal{R}) = \mathcal{L}(\mathcal{R}\hat{}\mathcal{R}_1) = \phi,\\ \mathcal{L}(\mathcal{R}_1 \oplus \mathcal{R}) = \mathcal{L}(\mathcal{R} \oplus \mathcal{R}_1) = \mathcal{L}(\mathcal{R}),\\ \mathcal{L}(\mathcal{R}_1^*) = \{\varepsilon\}, \text{ and } \mathcal{L}(\mathcal{R}_1, [a, b]) = \phi\,.\end{array}$$

Furthermore, for an HRE $\mathcal{R}$, it is not difficult to give an efficient algorithm for checking the emptiness of $\mathcal{R}$. Therefore, if $\mathcal{R}$ is not an empty HRE, we can find out an HRE $\mathcal{R}'$ efficiently such that there is no empty sub-expression in $\mathcal{R}'$ and that $\mathcal{L}(\mathcal{R}) = \mathcal{L}(\mathcal{R}')$. For the simplicity, from now on, unless otherwise stated, we assume that all HREs under consideration are not empty and do not have any empty sub-expression.

A *simple* HRE is an HRE in which there is no occurrence of the combinators $*$ (repetition) and $\oplus$ (union). From Definition 1, any simple HRE $\mathcal{R}$ can be rewritten as a simple HRE $\mathcal{R}'$ of the form $(v_1\hat{}v_2\hat{}\ldots\hat{}v_m, \Delta)$ such that $\mathcal{L}(\mathcal{R}) = \mathcal{L}(\mathcal{R}')$. For example, the simple HRE

$$(u\hat{}v, \{a \le c_1\lambda_1 + c_2\lambda_2 \le b\})\hat{}(s, \{d \le c_1\lambda_1 \le e\})$$

can be rewritten as

$$(u\hat{}v\hat{}s, \{a \le c_1\lambda_1 + c_2\lambda_2 + 0\lambda_3 \le b, d \le 0\lambda_1 + 0\lambda_2 + c_3\lambda_3 \le e\})\quad .$$

Therefore, from now on, we assume that any simple HRE is of the form

$$(v_1\hat{}v_2\hat{}\ldots\hat{}v_m, \Delta)\,,$$

where Δ is a finite set of linear inequalities of the form $a \le \sum_{i=1}^{m} c_i\lambda_i \le b$.

By a *normal form* we mean an HRE of the form

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_m\ ,$$

where $\mathcal{R}_j$s are simple HREs.

3 Checking Hybrid Regular Expressions for Linear Duration Invariants

As mentioned in the introduction of the paper, linear duration invariants form an important class of DC formulas for specifying the requirement of real-time and hybrid systems. A linear duration invariant $\mathcal{D}$ is of the form

$$T \ge \int 1 \ge t \Rightarrow \bigwedge_{j=1}^{k} (\sum_{i=1}^{n} c_{ij} \int S_i \le M_j)\,,$$

where T, t, c_{ij}, M_j are real numbers (T may be ∞), S_is are predicates over V.

For a location $v \in V$, for a predicate S over V, let $v \Rightarrow S$ denote that S holds during the system stays at v. For a timed sequence

$$\sigma = (v_1, t_1)\hat{}(v_2, t_2)\hat{} \ldots \hat{}(v_m, t_m) \ ,$$

the integrated duration of state S_i, i.e. the value of $\int S_i$, can be calculated as

$$\int S_i = \sum_{u \in \alpha_i} t_u \, ,$$

where $\alpha_i \hat{=} \{u \mid (1 \leq u \leq m) \wedge (v_u \Rightarrow S_i)\}$. Consequently,

$$\int 1 = \sum_{u=1}^{m} t_u \, ,$$

as $\{u \mid (1 \leq u \leq m) \wedge (v_u \Rightarrow 1)\} = \{1, 2, \ldots, m\}$.

Definition 3. *A time sequence* $\sigma = (v_1, t_1)\hat{}(v_2, t_2)\hat{} \ldots \hat{}(v_m, t_m)$ *over* V *satisfies a linear duration invariant* $\mathcal{D}$ *iff* $\bigwedge_{j=1}^{k}(\sum_{i=1}^{n} c_{ij}(\sum_{u \in \alpha_i} t_u) \leq M_j)$ *when* $T \geq \sum_{u=1}^{m} t_u \geq t$. *An HRE* $\mathcal{R}$ *satisfies a linear duration invariant* $\mathcal{D}$, *denoted by* $\mathcal{R} \models \mathcal{D}$, *iff any timed sequences* $\sigma \in \mathcal{L}(\mathcal{R})$ *satisfies* $\mathcal{D}$. □

In this section, we will give an algorithm for checking HREs for linear duration invariants.

3.1 Basic Idea

Let $\mathcal{R}$ be a simple HRE

$$\mathcal{R} = (\langle v_1, \lambda_1 \rangle \hat{} \langle v_2, \lambda_2 \rangle \hat{} \ldots \hat{} \langle v_m, \lambda_m \rangle, \Delta) \, .$$

From the definition of HREs, every $\sigma \in \mathcal{L}(\mathcal{R})$ is of the form

$$(v_1, t_1)\hat{}(v_2, t_2)\hat{} \ldots \hat{}(v_m, t_m) \, ,$$

where $t_1, t_2, \ldots, t_m$ satisfy the group of linear inequalities represented by Δ. Denoting this group of linear inequalities by C_1, the problem of checking $\mathcal{R} \models \mathcal{D}$ is then equivalent to the problem of finding the maximum value of the linear function

$$\sum_{i=1}^{n} c_{ij}(\sum_{u \in \alpha_i} t_u)$$

subject to the linear constraints C_1 and C_2 and checking whether it is not greater than M_j for all $j = 1, \ldots, k$, where C_2 denotes the inequality

$$t \leq t_1 + t_2 + \ldots + t_m \leq T \, .$$

The latters are linear programming problems.

Let $\mathcal{N} = \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_m$ be a normal form. Hence, each $\mathcal{R}_i$ $(1 \leq i \leq m)$ is a simple HRE. Since, by Definition 3,

$$\mathcal{N} \models D \Leftrightarrow \bigwedge_{i=1}^{m} \mathcal{R}_i \models D\,,$$

the problem of checking $\mathcal{N}$ for $\mathcal{D}$ can be solved by solving m linear programming problems $\mathcal{R}_i \models \mathcal{D},\ i = 1, 2, \ldots, m$.

Therefore, for a general HRE $\mathcal{R}$, for a linear duration invariant $\mathcal{D}$, if we can effectively find a normal form $\mathcal{N}$ such that $\mathcal{R} \models \mathcal{D}$ if and only if $\mathcal{N} \models \mathcal{D}$, then we can check $\mathcal{R} \models \mathcal{D}$ effectively. Based on this idea, in the following subsections, we will give an algorithm to check an HRE $\mathcal{R}$ for a linear duration invariant $\mathcal{D}$. Without loss of generality, throughout the following subsections, let $\mathcal{D}$ be

$$t \leq \int 1 \leq T \Rightarrow \sum_{i=1}^{n} c_i \int S_i \leq M\,,$$

and for any $\sigma = (v_1, t_1)\hat{}(v_2, t_2)\hat{} \ldots \hat{}(v_m, t_m) \in \mathcal{L}(\mathcal{R})$, let $\theta(\sigma, \mathcal{D})$ be the value of $\sum_{i=1}^{n} c_i \int S_i$ evaluated over σ,

$$\theta(\sigma, \mathcal{D}) = \sum_{i=1}^{n} c_i (\sum_{u \in \alpha_i} t_u)\,,$$

where $\alpha_i = \{u \mid (1 \leq u \leq m) \wedge (v_u \Rightarrow S_i)\}$.

For any simple HRE $\mathcal{R}$, let $M_\tau(\mathcal{R})$ $(m_\tau(\mathcal{R}))$ denote the supremum (infimum) of the set $\{\tau(\sigma) \mid \sigma \in \mathcal{L}(\mathcal{R})\}$. $M_\tau(\mathcal{R})$ $(m_\tau(\mathcal{R}))$ can be calculated by finding the maximal (minimal) value of the linear objective function $t_1 + t_2 + \ldots + t_m$ subject to the group of linear inequalities $C1$ associated with $\mathcal{R}$, which is a classical linear programming problem. If $m_\tau(\mathcal{R}) = 0$, $\mathcal{R}$ is said to be a *zero-simple* HRE; otherwise $\mathcal{R}$ is said to be a *nonzero-simple* HRE.

For any nonzero-simple HRE $\mathcal{R}$, let $M_\theta(\mathcal{R})$ denote the supremum of the set

$$\{\theta(\sigma, \mathcal{D}) | \sigma \in \mathcal{L}(\mathcal{R})\}\ .$$

Similarly to $M_\tau(\mathcal{R})$, $M_\theta(\mathcal{R})$ can be calculated effectively by finding the maximal value of the linear objective function $\sum_{i=1}^{n} c_i (\sum_{u \in \alpha_i} t_u)$ subject to the group of linear inequalities $C1$.

For a real number x, let $\lfloor x \rfloor$ denote the floor of x. For an HRE $\mathcal{R}$, let $\mathcal{R}^j$ denote the j-repetition of $\mathcal{R}$

$$\mathcal{R}^j = \underbrace{\mathcal{R}\hat{}\mathcal{R}\hat{} \ldots \hat{}\mathcal{R}}_{j},\ \mathcal{R}^0 = \varepsilon\,.$$

3.2 Foundations of the Model-Checking Algorithm

The basic idea of our algorithm for checking an HRE for a linear duration invariant $\mathcal{D}$, is to find out a normal form $\mathcal{N}$ such that $\mathcal{R} \models \mathcal{D}$ if and only if $\mathcal{N} \models \mathcal{D}$. In the following, we first introduce the concept *contexts* for describing

this idea formally, then give several lemmas and theorems as the foundations of the algorithm.

Let $\mathcal{R}$ be an HRE, and $\mathcal{R}_1$ be a sub-expression of $\mathcal{R}$. Replacing an occurrence of $\mathcal{R}_1$ in $\mathcal{R}$ with a letter X, we obtain a *context* of X. Any context $\mathcal{C}(X)$ of X, is associated with two real numbers $\varphi(\mathcal{C}(X))$ and $\omega(\mathcal{C}(X))$, which specify a lower bound and a upper bound of the constraints on the occurrence time enforced by the context on the variable X. If the context does not enforce any time constraint on X then $\varphi(\mathcal{C}(X)) = 0$ and $\omega(\mathcal{C}(X)) = \infty$.

Definition 4. *A context $\mathcal{C}(X)$ of X, $\varphi(\mathcal{C}(X))$ and $\omega(\mathcal{C}(X))$ are defined recursively as:*

1. *X is a context of X, and $\varphi(X) = 0$ and $\omega(X) = \infty$ (no additional constraint).*
2. *If $\mathcal{C}_1(X)$ is a context of X and $\mathcal{R}$ is an HRE, then $\mathcal{C}(X) = \mathcal{C}_1(X)\,\hat{}\,\mathcal{R}$ and $\mathcal{C}(X) = \mathcal{R}\,\hat{}\,\mathcal{C}_1(X)$ are contexts of X, and*
$$\varphi(\mathcal{C}(X)) = \varphi(\mathcal{C}_1(X))\,,\ \omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$$
(no additional constraint).
3. *If $\mathcal{C}_1(X)$ is a context of X and $\mathcal{R}$ is an HRE, then $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}$ and $\mathcal{C}(X) = \mathcal{R} \oplus \mathcal{C}_1(X)$ are contexts of X, and*
$$\varphi(\mathcal{C}(X)) = \varphi(\mathcal{C}_1(X))\,,\ \omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$$
(no additional constraint).
4. *If $\mathcal{C}_1(X)$ is a context of X, then $\mathcal{C}(X) = \mathcal{C}_1(X)^*$ is a context of X, and*
$$\varphi(\mathcal{C}(X)) = \varphi(\mathcal{C}_1(X))\,,\ \omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$$
(no additional constraint).
5. *If $\mathcal{C}_1(X)$ is a context of X, $a \in R^+, b \in R^+ \cup \{\infty\}, b > 0$, and $a \leq b$, then $\mathcal{C}(X) = (\mathcal{C}_1(X), [a, b])$ is a context of X, and*
$$\varphi(\mathcal{C}(X)) = \max(\varphi(\mathcal{C}_1(X)), a)\,,\ \omega(\mathcal{C}(X)) = \min(\omega(\mathcal{C}_1(X)), b)$$
(additional constraint enforced by $[a, b]$). □

For any context $\mathcal{C}(X)$, replacing X in $\mathcal{C}(X)$ with an HRE, say $\mathcal{R}$, we obtain an HRE, denoted by $\mathcal{C}(\mathcal{R})$.

A *finite* HRE is an HRE in which there is no occurrence of the combinator $*$ (repetition). By Definition 1, it is not difficult to prove that for any HRE $\mathcal{R}$, distributing $\hat{}$ over $\oplus$, and $[a, b]$ over $\oplus$, we obtain a normal form $\mathcal{R}'$ such that $\mathcal{L}(\mathcal{R}) = \mathcal{L}(\mathcal{R}')$. For example, for a finite HRE

$$\mathcal{R} = (((v_1\,\hat{}\,v_2, \Delta_1) \oplus (v_3\,\hat{}\,v_4, \Delta_2))\,\hat{}\,(v_5\,\hat{}\,v_6, \Delta_3), [a, b])\,,$$

distributing $\hat{}$ over $\oplus$, and $[a, b]$ over $\oplus$, we get a normal form

$$\mathcal{R}' = ((v_1\,\hat{}\,v_2, \Delta_1)\,\hat{}\,(v_5\,\hat{}\,v_6, \Delta_3), [a, b]) \oplus ((v_3\,\hat{}\,v_4, \Delta_1)\,\hat{}\,(v_5\,\hat{}\,v_6, \Delta_3), [a, b])\,,$$

such that $\mathcal{L}(\mathcal{R}) = \mathcal{L}(\mathcal{R}')$. Hence, for an HRE $\mathcal{R}$ and a linear duration invariant $\mathcal{D}$, we attempt to find a normal form $\mathcal{N}$ such that $\mathcal{L}(\mathcal{R}) \models \mathcal{D}$ if and only if $\mathcal{L}(\mathcal{N}) \models \mathcal{D}$ by the following procedure:

Step 0. Let $\mathcal{R}' := \mathcal{R}$.
Step 1. For $\mathcal{R}'$, distributing $\hat{}$ over $\oplus$, and $[a, b]$ over $\oplus$, we obtain $\mathcal{Q}$. If $\mathcal{Q}$ is a normal form, then we are done.
Step 2. For a sub-expression $\mathcal{Q}_S$ of $\mathcal{Q}$ which is of the form $\mathcal{Q}_S = {\mathcal{Q}_1}^*$, replacing an occurrence of $\mathcal{Q}_S$ in $\mathcal{Q}$ with X, we obtain a context $\mathcal{C}_\mathcal{Q}(X)$ such that $\mathcal{R}' = \mathcal{C}_\mathcal{Q}(\mathcal{Q}_S)$.
Step 3. Finding a finite HRE $\mathcal{Q}'_S$ such that $\mathcal{C}_\mathcal{Q}(\mathcal{Q}_S) \models \mathcal{D}$ iff $\mathcal{C}_\mathcal{Q}(\mathcal{Q}'_S) \models \mathcal{D}$. Let $\mathcal{R}' := \mathcal{C}_\mathcal{Q}(\mathcal{Q}'_S)$, and go to Step 1. □

Obviously the procedure is correct. The problem is how to find $\mathcal{Q}'_S$ in Step 3. The following lemmas and theorems will help to solve that problem.

Let $\mathcal{C}(X)$ be a context.

Lemma 1. *(1.)* Let $\mathcal{R}$ and $\mathcal{R}'$ be HREs. If for any $\sigma \in \mathcal{L}(\mathcal{R})$, there is $\sigma' \in \mathcal{L}(\mathcal{R}')$ such that $\tau(\sigma) = \tau(\sigma')$ and $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$, then $\mathcal{C}(\mathcal{R}') \models \mathcal{D}$ implies $\mathcal{C}(\mathcal{R}) \models \mathcal{D}$. □

Lemma 2. *(2.)* Suppose $\omega(\mathcal{C}(X)) = \infty$, and $\mathcal{R}$ be a nonzero-simple HRE $\mathcal{R}$ such that $M_\theta(\mathcal{R}) \leq 0$. Then for any real number N_t, for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ such that $\tau(\sigma) \geq N_t$, there is $\sigma' \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j)$ such that

$$\tau(\sigma') \geq N_t \ \text{ and } \ \theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D}),$$

where $p = (\lfloor h/m_\tau(\mathcal{R}) \rfloor + 1)$, and $h = \max(\varphi(\mathcal{C}(X), N_t)$. □

Lemma 3. *(3.)* Suppose $\omega(\mathcal{C}(X)) = \infty$, and $\mathcal{R}$ be a nonzero-simple HRE such that $M_\theta(\mathcal{R}) > 0$. Then for any nonnegative real numbers N_t and M_r, there is $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ such that $\tau(\sigma) \geq N_t$ and $\theta(\sigma, \mathcal{D}) > M_r$. □

Lemma 4. *(4.)* Suppose $\mathcal{R}$ be a nonzero-simple HRE, and $T \neq \infty$. Then for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, $\tau(\sigma) \leq T$ implies $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, where $p = \lfloor T/m_\tau(\mathcal{R}) \rfloor + 1$. □

Lemma 5. *(5.)*Suppose $\mathcal{R}$ be a nonzero-simple HRE, $a \in R^+$, $b \in R^+ \cup \{\infty\}, a \leq b$, and $b > 0$. Then

$$\mathcal{L}((\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a, b])) \supseteq \mathcal{L}((\mathcal{C}(\mathcal{R}^*), [a, b])),$$

where $p = \lfloor b/m_\tau(\mathcal{R}) \rfloor + 1$. □

Lemma 6. *(6.)* Let $\mathcal{R}$ be a nonzero-simple HRE. Then $\mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j)) \supseteq \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, where $p = \lfloor \omega(\mathcal{C}(X))/m_\tau(\mathcal{R}) \rfloor + 1$. □

These lemmas can be proved by induction on the structure of context, and their detailed proofs are presented in the appendix. From these lemmas, we can prove the following theorems.

Theorem 1. *(1.)*Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be HREs. Then

$$\mathcal{C}((\mathcal{R}_1 \oplus \mathcal{R}_2)^*) \models \mathcal{D} \ \text{ iff } \ \mathcal{C}(({\mathcal{R}_1}^*)\hat{}({\mathcal{R}_2}^*)) \models \mathcal{D}.$$

Proof. By Definition 1, $\mathcal{L}((\mathcal{R}_1{}^*)^\frown(\mathcal{R}_2{}^*)) \subseteq \mathcal{L}((\mathcal{R}_1 \oplus \mathcal{R}_2)^*)$. From Lemma 1, the half of the claim follows, i.e.

$$\mathcal{C}((\mathcal{R}_1 \oplus \mathcal{R}_2)^*) \models \mathcal{D} \;\text{ implies }\; \mathcal{C}((\mathcal{R}_1{}^*)^\frown(\mathcal{R}_2{}^*)) \models \mathcal{D}\,,$$

The other half can be proved as follows. For any $\sigma_1 \in \mathcal{L}(\mathcal{R}_1)$ and $\sigma_2 \in \mathcal{L}(\mathcal{R}_2)$, since $\tau(\sigma_1{}^\frown\sigma_2) = \tau(\sigma_1) + \tau(\sigma_2)$ and $\theta(\sigma_1{}^\frown\sigma_2, \mathcal{D}) = \theta(\sigma_1, \mathcal{D}) + \theta(\sigma_2, \mathcal{D})$, we have $\tau(\sigma_1{}^\frown\sigma_2) = \tau(\sigma_2{}^\frown\sigma_1)$ and $\theta(\sigma_1{}^\frown\sigma_2, \mathcal{D}) = \theta(\sigma_2{}^\frown\sigma_1, \mathcal{D})$. Therefore, any $\sigma \in \mathcal{L}(\mathcal{C}((\mathcal{R}_1 \oplus \mathcal{R}_2)^*)$ can be permuted into $\sigma' \in \mathcal{L}(\mathcal{C}((\mathcal{R}_1{}^*)^\frown(\mathcal{R}_2{}^*)))$. Hence, from Lemma 1, the result follows. □

Theorem 2. *(2.)*Let $\mathcal{R}$ be a zero-simple HRE, $\mathcal{R} = (v_1{}^\frown v_2{}^\frown \ldots {}^\frown v_m, \Delta)$. Let Δ' be the set $\{0 \le \sum_{i=1}^m c_i\lambda_i \mid 0 \le \sum_{i=1}^m c_i\lambda_i \le b \in \Delta \wedge \exists j \cdot (1 \le j \le m \wedge c_j < 0)\}$, and $\mathcal{R}' = (v_1{}^\frown v_2{}^\frown \ldots {}^\frown v_m, \Delta')$. Then $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ iff $\mathcal{C}(\mathcal{R}') \models \mathcal{D}$.

Proof. Before the proof, we should note that by the definition of zero-simple HREs, $\tau(\mathcal{R}) = 0$ implies that for any inequality $a \le c_1\lambda_1 + c_2\lambda_2 + \ldots + c_m\lambda_m \le b$ in Δ, $a \le 0$ and $b \ge 0$.

The half of the claim that $\mathcal{C}(\mathcal{R}') \models \mathcal{D}$ implies $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$, is explained as follows. By Definition 1, any $\sigma \in \mathcal{L}(\mathcal{R}^*)$ is of the form $\sigma_1{}^\frown\sigma_2{}^\frown \ldots {}^\frown\sigma_n$, where

$$\sigma_i = (v_1, t_{i1})^\frown(v_2, t_{i2})^\frown \ldots {}^\frown(v_m, t_{im}) \;\in\; \mathcal{L}(\mathcal{A}) \;\; (i = 1, 2, \ldots, n)\,.$$

For any j $(1 \le j \le m)$, let $t'_j = t_{1j} + t_{2j} + \ldots + t_{nj}$, and let

$$\sigma' = (v_1, t'_1)^\frown(v_2, t'_2)^\frown \ldots {}^\frown(v_m, t'_m)\,.$$

Since for any i $(1 \le i \le n)$, $t_{i1}, t_{i2}, \ldots, t_{im}$ satisfy Δ, $t'_1, t'_2, \ldots, t'_m$ satisfy Δ' as well. It follows that $\sigma' \in \mathcal{L}(\mathcal{R}')$. Since $\theta(\sigma, \mathcal{D}) = \theta(\sigma', \mathcal{D})$ and $\tau(\sigma) = \tau(\sigma')$, the first half of the claim follows from Lemma 1.

The other half of the claim, i.e. $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ implies $\mathcal{C}(\mathcal{R}') \models \mathcal{D}$, can be proved as follows. For any $\sigma' = (v_1, t_1)^\frown(v_2, t_2)^\frown \ldots {}^\frown(v_m, t_m) \in \mathcal{L}(\mathcal{R}')$, since $t_1, t_2, \ldots, t_m$ satisfy Δ', for any $0 \le \sum_{i=1}^m c_i\lambda_i \le b \in \Delta$, we have $\sum_{i=1}^m c_i t_i \ge 0$. Because for each inequality $a \le c_1\lambda_1 + c_2\lambda_2 + \ldots + c_m\lambda_m \le b$ in Δ, $a \le 0$ and $b \ge 0$, and because Δ is a finite set, we can choose a natural number p such that for any inequality $a \le c_1\lambda_1 + c_2\lambda_2 + \ldots + c_m\lambda_m \le b \in \Delta$,

$$a \le \frac{c_1t_1 + c_2t_2 + \ldots + c_mt_m}{p} \le b\,.$$

For each i $(1 \le i \le m)$, let $b_i = t_i/p$, and let $\sigma_b = (v_1, b_1)^\frown(v_2, b_2)^\frown \ldots {}^\frown(v_m, b_m)$. Obviously, $\sigma \in \mathcal{L}(\mathcal{R})$. Let

$$\sigma = \underbrace{\sigma_b{}^\frown\sigma_b{}^\frown \ldots {}^\frown\sigma_b}_{p}\,.$$

It follows that $\sigma \in \mathcal{L}(\mathcal{R}^*)$. Since $\theta(\sigma, \mathcal{D}) = \theta(\sigma', \mathcal{D})$ and $\tau(\sigma) = \tau(\sigma')$, by Lemma 1, $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ implies $\mathcal{C}(\mathcal{R}') \models \mathcal{D}$. □

Theorem 3. *(3.)*Suppose $\omega(\mathcal{C}(X)) = \infty$, $T = \infty$, and $\mathcal{R}$ be a nonzero-simple HRE such that $M_\theta(\mathcal{R}) > 0$. Then $\mathcal{C}(\mathcal{R}^*) \not\models \mathcal{D}$.

Proof. The theorem follows immediately from Lemma 3. □

Theorem 4. *(4.)*Suppose $\omega(\mathcal{C}(X)) = \infty$, $T = \infty$, and $\mathcal{R}$ be a nonzero-simple HRE such that $M_\theta(\mathcal{R}) \leq 0$. Then $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ iff $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) \models \mathcal{D}$, where $p = (\lfloor h/m_\tau(\mathcal{R})\rfloor + 1), h = \max(\varphi(\mathcal{C}(X), t)$.

Proof. By Definition 1, $\mathcal{L}(\mathcal{R}^*) \supseteq \mathcal{L}(\oplus_{j=0}^{p}\mathcal{R}^j)$ holds, which by Lemma 1 implies a half of the claim, i.e. $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ implies $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) \models \mathcal{D}$. The other half is straightforward from Lemma 2. □

Theorem 5. *(5.)*Suppose $\omega(\mathcal{C}(X)) \neq \infty$ or $T \neq \infty$, and $\mathcal{R}$ be a nonzero-simple HRE. Then $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ iff $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) \models \mathcal{D}$, where $p = (\lfloor h/m_\tau(\mathcal{R})\rfloor + 1)$, $h = \min(\omega(\mathcal{C}(X), T)$.

Proof. One half of the claim, i.e. $\mathcal{C}(\mathcal{R}^*) \models \mathcal{D}$ implies $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) \models \mathcal{D}$ is exactly the same as the proof of Theorem 4. The other half of the claim is a direct consequence of Lemmas 4 and 6. □

3.3 The Model-Checking Algorithm

Based on the theorems given in section 3.1, the algorithm to check an HRE $\mathcal{R}$ for a linear duration invariant $\mathcal{D}$ is now described as follows.

Step 0. Let $\mathcal{R}' := \mathcal{R}$.
Step 1. For $\mathcal{R}'$, distributing $\hat{}$ over $\oplus$, and $[a, b]$ over $\oplus$, we obtain $\mathcal{Q}$.
Step 2. Finding a sub-expression $\mathcal{Q}_S$ of $\mathcal{Q}$ which has one of the following three forms:
1. $\mathcal{Q}_S = (\mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \ldots \oplus \mathcal{R}_k)^*$ $(k \geq 2)$, where every $\mathcal{R}_i$ $(1 \leq i \leq m)$ is a simple HRE.
2. $\mathcal{Q}_S = \mathcal{R}_1^*$, where $\mathcal{R}_1$ is a nonzero-simple HRE.
3. $\mathcal{Q}_S = \mathcal{R}_1^*$, where $\mathcal{R}_1$ is a zero-simple HRE.

If such $\mathcal{Q}_S$ could not be found, goto Step 6 (note that it is not difficult to prove that if we can not find out such a $\mathcal{Q}_S$, then $\mathcal{Q}$ is a normal form); otherwise replacing the occurrence of $\mathcal{Q}_S$ in $\mathcal{Q}$ with X, we get a context $\mathcal{C}_Q(X)$ such that $\mathcal{Q} = \mathcal{C}_Q(\mathcal{Q}_S)$. Then, if $\mathcal{Q}_S$ has the first form, goto Step 3; if $\mathcal{Q}_S$ has second form, goto Step 4; if $\mathcal{Q}_S$ has the third form, goto Step 5.
Step 3. By Theorem 2, we transform $\mathcal{Q}$ into $\mathcal{Q}' = \mathcal{C}_Q((\mathcal{R}_1)^*\hat{}(\mathcal{R}_2)^*\hat{}\ldots\hat{}(\mathcal{R}_m)^*)$. Thus, let $\mathcal{R}' := \mathcal{Q}'$, and goto Step 1.
Step 4. We first calculate $\omega(\mathcal{C}_Q(X))$ and $M_\theta(\mathcal{R}_1)$. If $\omega(\mathcal{C}_Q(X)) \neq \infty$ or $T \neq \infty$, then by Theorem 5, we transform $\mathcal{Q}$ into $\mathcal{Q}' = \mathcal{C}_Q(\oplus_{j=0}^{p}\mathcal{R}_1^j)$, where $p = (\lfloor h/m_\tau(\mathcal{R}_1)\rfloor + 1)$, and $h = \min(\omega(\mathcal{C}_Q(X), T)$. Therefore, let $\mathcal{R}' := \mathcal{Q}'$, and goto Step 1.
Otherwise, $\omega(\mathcal{C}_Q(X)) = \infty$ and $T = \infty$. If $M_\theta(\mathcal{R}_1) > 0$, then by Theorem 3, we conclude $\mathcal{C}_Q(\mathcal{R}_1^*) \not\models \mathcal{D}$ and exit. Otherwise, by Theorem 4, we transform $\mathcal{Q}$ into $\mathcal{Q}' = \mathcal{C}_Q(\oplus_{j=0}^{p}\mathcal{R}_1^j)$, where $p = (\lfloor h/m_\tau(\mathcal{R}_1)\rfloor + 1)$, and $h = \max(\varphi(\mathcal{C}_Q(X), t)$. Let $\mathcal{R}' := \mathcal{Q}'$, and goto Step 1.

Step 5. By Theorem 2, we transform $\mathcal{Q}$ into $\mathcal{Q}' = \mathcal{C}_Q(\mathcal{R}_1')$, where $\mathcal{R}_1'$ is the simple HRE defined from $\mathcal{R}_1$ in Theorem 2. Let $\mathcal{R}' := \mathcal{Q}'$, and goto Step 1.
Step 6. Since Q is a normal form now, we check $\mathcal{Q} \models \mathcal{D}$ by linear programming. If $\mathcal{Q} \models \mathcal{D}$, then $\mathcal{R} \models \mathcal{D}$; otherwise $\mathcal{R} \not\models \mathcal{D}$.

□

To illustrate how the algorithm works, we apply it to check the water level monitor for its requirement. The behaviour of the water level monitor is represented by the HRE $\mathcal{R}_w$ given in Section 2:

$$\begin{aligned}\mathcal{R}_w = \varepsilon \oplus (s_1, [0,9]) \oplus (s_1, [9,9])\hat{}(s_2, [0,2]) \oplus \mathcal{R}_1 \\ \oplus \mathcal{R}_2\hat{}\mathcal{R}_3^*\hat{}((s_4, [0,2]) \oplus \mathcal{R}_4 \oplus \mathcal{R}_5 \oplus \mathcal{R}_6)\end{aligned}$$

where

$$\begin{aligned}\mathcal{R}_1 &= (s_1\hat{}s_2\hat{}s_3, \{\lambda_1 = 9,\ \lambda_2 = 2,\ 2\lambda_3 - \lambda_2 \le 5\}) \\ \mathcal{R}_2 &= (s_1\hat{}s_2\hat{}s_3, \{\lambda_1 = 9,\ \lambda_2 = 2,\ 2\lambda_3 - \lambda_2 = 5\}) \\ \mathcal{R}_3 &= (s_4\hat{}s_1\hat{}s_2\hat{}s_3, \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 = 5,\ \lambda_3 = 2,\ 2\lambda_4 - \lambda_3 = 5\}) \\ \mathcal{R}_4 &= (s_4\hat{}s_1, \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 \le 5\}) \\ \mathcal{R}_5 &= (s_4\hat{}s_1\hat{}s_2, \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 = 5,\ 0 \le \lambda_3 \le 2\}) \\ \mathcal{R}_6 &= (s_4\hat{}s_1\hat{}s_2\hat{}s_3, \\ &\qquad \{\lambda_1 = 2,\ \lambda_2 - 2\lambda_1 = 5,\ \lambda_3 = 2,\ -5 \le \lambda_3 - 2\lambda_4\})\,.\end{aligned}$$

and the requirement of the system is represented by the linear duration invariants given in the introduction of the paper:

$$\begin{aligned}0 \le \textstyle\int 1 \le \infty \Rightarrow \int s_1 + \int s_2 - 2\int s_3 - 2\int s_4 \le 11 \\ 0 \le \textstyle\int 1 \le \infty \Rightarrow -\int s_1 - \int s_2 + 2\int s_3 + 2\int s_4 \le 0\,.\end{aligned}$$

Let $\mathcal{R}_w = \mathcal{R}_{w1} \oplus \mathcal{R}_{w2}$, where $\mathcal{R}_{w1} = \varepsilon \oplus (s_1, [0,9]) \oplus (s_1, [9,9])\hat{}(s_2, [0,2]) \oplus \mathcal{R}_1$ and $\mathcal{R}_{w2} = \mathcal{R}_2\hat{}\mathcal{R}_3^*\hat{}((s_4, [0,2]) \oplus \mathcal{R}_4 \oplus \mathcal{R}_5 \oplus \mathcal{R}_6)$. Since for a linear duration invariant $\mathcal{D}$, $\mathcal{R}_w \models \mathcal{D}$ if and only if $\mathcal{R}_{w1} \models \mathcal{D}$ and $\mathcal{R}_{w2} \models \mathcal{D}$, we can solve the problem by checking $\mathcal{R}_{w1}$ and $\mathcal{R}_{w2}$ for the requirement separately. Because $\mathcal{R}_{w1}$ is a simple HRE, checking $\mathcal{R}_{w1}$ for the requirement is easy and is omitted here.

In the following, let us check $\mathcal{R}_{w2}$ for the linear duration invariant

$$0 \le \textstyle\int 1 \le \infty \Rightarrow \int s_1 + \int s_2 - 2\int s_3 - 2\int s_4 \le 11\,.$$

Starting from Step 0, let $\mathcal{R}' := \mathcal{R}_{w2}$. At Step 1, for $\mathcal{R}'$, distributing $\hat{}$ over $\oplus$ and $[a,b]$ over $\oplus$, we obtain $\mathcal{Q} = \mathcal{R}'$. At Step 2, choose $\mathcal{Q}_S = \mathcal{R}_3^*$, and

$$\mathcal{C}_Q(X) = \mathcal{R}_2\hat{}X\hat{}((s_4, [0,2]) \oplus \mathcal{R}_4 \oplus \mathcal{R}_5 \oplus \mathcal{R}_6)\,.$$

Since $\mathcal{Q}_S$ has the 2th form, Step 4 should be the following one. By calculating $\omega(\mathcal{C}_Q(X))$ and $M_\theta(\mathcal{R}_3)$ we found that $\omega(\mathcal{C}_Q(X)) = \infty$, $T = \infty$, and $M_\theta(\mathcal{A}) \le 0$. By Theorem 4, we transform $\mathcal{Q}$ into $\mathcal{Q}' = \mathcal{C}_Q(\oplus_{j=0}^{p}\mathcal{R}_3^j)$, where p is defined in Theorem 4. By a trivial calculation, $p = 1$ and $\mathcal{Q}' = \mathcal{C}_Q(\varepsilon \oplus \mathcal{R}_3)$. Let

$$\mathcal{R}' := \mathcal{R}_2\hat{}(\varepsilon \oplus \mathcal{R}_3)\hat{}((s_4, [0,2]) \oplus \mathcal{R}_4 \oplus \mathcal{R}_5 \oplus \mathcal{R}_6)\,,$$

and goto Step 1 for repetition. At Step 1, for $\mathcal{R}'$, distributing $\hat{}$ over $\oplus$ and $[a, b]$ over $\oplus$, we obtain

$$\begin{aligned}\mathcal{Q} = \mathcal{R}_2\hat{}(s_4, [0,2]) \oplus \mathcal{R}_2\hat{}\mathcal{R}_4 \oplus \mathcal{R}_2\hat{}\mathcal{R}_5 \oplus \mathcal{R}_2\hat{}\mathcal{R}_6 \oplus \mathcal{R}_2\hat{}\mathcal{R}_3\hat{}(s_4, [0,2]) \\ \oplus \mathcal{R}_2\hat{}\mathcal{R}_3\hat{}\mathcal{R}_4 \oplus \mathcal{R}_2\hat{}\mathcal{R}_3\hat{}\mathcal{R}_5 \oplus \mathcal{R}_2\hat{}\mathcal{R}_3\hat{}\mathcal{R}_6\,.\end{aligned}$$

At Step 2, Since $\mathcal{Q}$ is a normal form, $\mathcal{Q}_S$ cannot be found, and hence, Step 8 is taken as the last one. To perform Step 8, we have to solve the following eight linear programming problems.

Pr. No	Objective function	Constraints
1	$t_1 + t_2 - 2t_3 - 2t_4$	$t_1 = 9,\ t_2 = 2,$ $t_3 = 3.5,\ 0 \le t_4 \le 2$
2	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ 0 \le t_5 \le 9$
3	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5 + t_6$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ t_5 = 9,\ 0 \le t_6 \le 2$
4	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5 + t_6 - 2t_7$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ t_5 = 9,\ t_6 = 2,$ $0 \le t_7 \le 3.5$
5	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5 + t_6 - 2t_7 - 2t_8$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ t_5 = 9,\ t_6 = 2,$ $t_7 = 3.5,\ 0 \le t_8 \le 2$
6	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5 + t_6 - 2t_7 - 2t_8$ $+t_9$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ t_5 = 9,\ t_6 = 2,$ $t_7 = 3.5,\ t_8 = 2,\ 0 \le t_9 \le 9$
7	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5 + t_6 - 2t_7 - 2t_8$ $+t_9 + t_{10}$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ t_5 = 9,\ t_6 = 2,\ t_7 = 3.5,$ $t_8 = 2,\ t_9 = 9,\ 0 \le t_{10} \le 2$
8	$t_1 + t_2 - 2t_3 - 2t_4$ $+t_5 + t_6 - 2t_7 - 2t_8$ $+t_9 + t_{10} - 2t_{11}$	$t_1 = 9,\ t_2 = 2,\ t_3 = 3.5,$ $t_4 = 2,\ t_5 = 9,\ t_6 = 2,$ $t_7 = 3.5,\ t_8 = 2,\ t_9 = 9,$ $t_{10} = 2,\ 0 \le t_{11} \le 3.5$

We have solved these linear problems and found that for each one, the maximal value of the objective function is not greater than 11. Thus, we can conclude that one part of the requirement is satisfied by the system.

Similarly, we can check for the other part of the requirement, and conclude that the water-level monitor keeps the water-level between 1 and 12 inches.

4 Conclusion

We have presented our algorithm for checking a hybrid system whose behaviour is represented by an HRE, for a linear duration invariant. Our algorithm transforms the problem into a finite number of linear programming problems. Since the resulting linear programming problems are independent from each others, and since the programming problems can be solved efficiently, our algorithm when it can be applied gives an efficient way to solve the model checking problem.

The model checking problem for hybrid real time systems in general is very difficult. Even for a well-formed class of hybrid systems – the class of linear hybrid automata – the problem is still undecidable in general. HREs define a subclass of linear hybrid automata for which the problem can be solved efficiently. It should be noticed that the subclass defined by HREs is not comparable with the class of linear hybrid systems defined in [6]. Thus, theoretically the paper gives a new result for the decidability of the model checking problem.

Currently, a model checking tool based on the algorithms presented in this paper and in our previous ones is being developed. Hopefully, the tool will display more clearly the advantages of our approach.

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A Proof of Lemmas

In this appendix, we present the proof of Lemmas $1-6$. These lemmas will be proved by induction on the structure of context. Since the proof for the structures $\mathcal{R} \oplus \mathcal{C}_1(X)$ and $\mathcal{R}\hat{}\mathcal{C}_1(X)$ is similar to the one of the structures $\mathcal{C}_1(X) \oplus \mathcal{R}$ and $\mathcal{C}_1(X)\hat{}\mathcal{R}$, it is left for the reader.

Lemma 7. *(1.)* Let $\mathcal{R}$ and $\mathcal{R}'$ be HREs. If for any $\sigma \in \mathcal{L}(\mathcal{R})$, there is $\sigma' \in \mathcal{L}(\mathcal{R}')$ such that $\tau(\sigma) = \tau(\sigma')$ and then $\mathcal{C}(\mathcal{R}') \models \mathcal{D}$ implies $\mathcal{C}(\mathcal{R}) \models \mathcal{D}$.

Proof. Lemma 1 follows immediately from the following claim: if for any $\sigma_1 \in \mathcal{L}(\mathcal{R})$, there is $\sigma_1' \in \mathcal{L}(\mathcal{R}')$ such that $\tau(\sigma_1) = \tau(\sigma_1')$ and $\theta(\sigma_1, \mathcal{D}) \leq \theta(\sigma_1', \mathcal{D})$, then for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}))$, there is $\sigma' \in \mathcal{L}(\mathcal{C}(\mathcal{R}'))$ such that $\tau(\sigma) = \tau(\sigma')$ and $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$. We prove the claim by induction on the structure of context.

- **Basic Case:** Let $\mathcal{C}(X) = X$. Then $\mathcal{C}(\mathcal{R}) = \mathcal{R}$ and $\mathcal{C}(\mathcal{R}') = \mathcal{R}'$. By assumption, the basic case holds.
- **Induction Step:** Assume that the claim holds for a context $\mathcal{C}_1(X)$, and let $\mathcal{C}(X)$ be defined from $\mathcal{C}_1(X)$.
  1. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then $\mathcal{C}(\mathcal{R}) = \mathcal{C}_1(\mathcal{R}) \oplus \mathcal{R}_1$ and $\mathcal{C}(\mathcal{R}') = \mathcal{C}_1(\mathcal{R}') \oplus \mathcal{R}_1$. Since, by Definition 1,

     $$\mathcal{L}(\mathcal{C}(\mathcal{R})) = \mathcal{L}(\mathcal{C}_1(\mathcal{R})) \cup \mathcal{L}(\mathcal{R}_1) \;\text{ and }\; \mathcal{L}(\mathcal{C}(\mathcal{R}')) = \mathcal{L}(\mathcal{C}_1(\mathcal{R}')) \cup \mathcal{L}(\mathcal{R}_1)\,,$$

     by the inductive hypothesis, the claim holds.
  2. Let $\mathcal{C}(X) = \mathcal{C}_1(X)\hat{}\mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

     $$\mathcal{C}(\mathcal{R}) = \mathcal{C}_1(\mathcal{R})\hat{}\mathcal{R}_1 \;\text{ and }\; \mathcal{C}(\mathcal{R}') = \mathcal{C}_1(\mathcal{R}')\hat{}\mathcal{R}_1\,.$$

     For any $\sigma = \sigma_1\hat{}\sigma_{\mathcal{R}_1} \in \mathcal{L}(\mathcal{C}(\mathcal{R}))$, where $\sigma_1 \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}))$ and $\sigma_{\mathcal{R}1} \in \mathcal{L}(\mathcal{R}_1)$, by the inductive hypothesis, there is $\sigma_1' \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}'))$ such that

     $$\tau(\sigma_1) = \tau(\sigma'_1) \;\text{ and }\; \theta(\sigma_1, \mathcal{D}) \leq \theta(\sigma'_1, \mathcal{D})\,.$$

     Let $\sigma' = \sigma_1'\hat{}\sigma_{\mathcal{R}_1}$. It follows that $\sigma' \in \mathcal{L}(\mathcal{C}(\mathcal{R}'))$. Since

     $$\begin{aligned}&\tau(\sigma') = \tau(\sigma_1') + \tau(\sigma_{\mathcal{R}_1}),\ \text{and}\\ &\theta(\sigma', \mathcal{D}) = \theta(\sigma_1', \mathcal{D}) + \theta(\sigma_{\mathcal{R}_1}, \mathcal{D})\,,\end{aligned}$$

     we have $\tau(\sigma) = \tau(\sigma')$ and $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$, i.e. the claim holds.
  3. Let $\mathcal{C}(X) = \mathcal{C}_1(X)^*$. Then $\mathcal{C}(\mathcal{R}) = \mathcal{C}_1(\mathcal{R})^*$ and $\mathcal{C}(\mathcal{R}') = \mathcal{C}_1(\mathcal{R}')^*$. For any

     $$\sigma = \sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m \in \mathcal{L}(\mathcal{C}(\mathcal{R}))\,,$$

     where each $\sigma_i \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}))$ $(1 \leq i \leq m)$, by the inductive hypothesis, for each i $(1 \leq i \leq m)$, there is $\sigma_i' \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}'))$ such that $\tau(\sigma_i) = \tau(\sigma'_i)$ and $\theta(\sigma_i, \mathcal{D}) \leq \theta(\sigma'_i, \mathcal{D})$. Let $\sigma' = \sigma_1'\hat{}\sigma_2'\hat{}\ldots\hat{}\sigma_m' \in \mathcal{L}(\mathcal{C}(\mathcal{R}'))$. Similarly the previous case, we have $\tau(\sigma) = \tau(\sigma')$ and $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$, i.e. the claim holds.

4. Let $\mathcal{C}(X) = (\mathcal{C}_1(X), [a, b])$ where $a \in R^+, b \in R^+, b \geq 0$, and $a \leq b$. Then $\mathcal{C}(\mathcal{R}) = (\mathcal{C}_1(\mathcal{R}), [a, b])$ and $\mathcal{C}(\mathcal{R}') = (\mathcal{C}_1(\mathcal{R}'), [a, b])$. For any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}))$, since, by Definition 1, $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}))$, by the inductive hypothesis, there is $\sigma' \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}'))$ such that $\tau(\sigma) = \tau(\sigma')$ and $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$, which implies that $\sigma' \in \mathcal{L}(\mathcal{C}(\mathcal{R}'))$, i.e. the claim holds. □

Lemma 8. *(2.)* Suppose $\omega(\mathcal{C}(X)) = \infty$, and $\mathcal{R}$ be a nonzero-simple HRE $\mathcal{R}$ such that $M_\theta(\mathcal{R}) \leq 0$. Then for any real number N_t, for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ such that $\tau(\sigma) \geq N_t$, there is $\sigma' \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p} \mathcal{R}^j)$ such that

$$\tau(\sigma') \geq N_t \;\text{ and }\; \theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D}),$$

where $p = (\lfloor h/m_\tau(\mathcal{R}) \rfloor + 1)$, and $h = \max(\varphi(\mathcal{C}(X), N_t)$.

Proof. The proof goes by induction on the structure of context.

- **Basic Case:** Let $\mathcal{C}(X) = X$. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{R}^*$ and $\mathcal{C}(\oplus_{j=0}^{p} \mathcal{R}^j) = \oplus_{j=0}^{p} \mathcal{R}^j$. By Definition 1, any $\sigma \in \mathcal{L}(\mathcal{R}^*)$ $(\tau(\sigma) \geq N_t)$ is of the form $\sigma_1 \hat{} \sigma_2 \hat{} \ldots \hat{} \sigma_m$ where for each i $(1 \leq i \leq m)$, $\sigma_i \in \mathcal{L}(\mathcal{R})$. If $m \leq p$, then $\sigma \in \mathcal{L}(\oplus_{j=0}^{p} \mathcal{A}^j)$, i.e. the lemma holds for basic case. If $m > p$, let $\sigma' = \sigma_1 \hat{} \sigma_2 \hat{} \ldots \hat{} \sigma_p$. It follows that $\sigma' \in \mathcal{L}(\oplus_{j=0}^{p} \mathcal{R}^j)$. Since, by assumption, $p = (\lfloor h/m_\tau(\mathcal{R}) \rfloor + 1)$ and $h = \max(\varphi(\mathcal{C}(X), N_t)$, it follows that $\tau(\sigma') \geq N_t$. Because $M_\theta(\mathcal{R}) \leq 0$, it is obvious that $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$, i.e. the basic case holds.
- **Induction Step:** Assume that the claim holds for a context $\mathcal{C}_1(X)$, and let $\mathcal{C}(X)$ be defined from $\mathcal{C}_1(X)$.
  1. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \hat{} \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

     $$\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*) \hat{} \mathcal{R}_1 \;\text{ and }\; \mathcal{C}(\oplus_{j=0}^{p} \mathcal{R}^j) = \mathcal{C}_1(\oplus_{j=0}^{p} \mathcal{R}^j) \hat{} \mathcal{R}_1 .$$

     From Definition 4, it follows that $\omega(\mathcal{C}_1(X)) = \omega(\mathcal{C}(X))$. Since by the assumption, $\omega(\mathcal{C}(X)) = \infty$, we have $\omega(\mathcal{C}_1(X)) = \infty$. On the other hand, for any

     $$\sigma = \sigma_1 \hat{} \sigma_{\mathcal{R}} \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*)) \;\; (\sigma_1 \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*)),\; \sigma_{\mathcal{R}} \in \mathcal{L}(\mathcal{R}_1)),$$

     we have $\tau(\sigma) = \tau(\sigma_1) + \tau(\sigma_{\mathcal{R}}) \geq N_t$. It follows that $\tau(\sigma_1) \geq N_t - \tau(\sigma_{\mathcal{R}})$. By the inductive hypothesis, there is $\sigma_1' \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p_1} \mathcal{R}^j))$ such that

     $$\tau(\sigma_1') \geq N_t - \tau(\sigma_{\mathcal{R}}) \;\text{ and }\; \theta(\sigma_1, \mathcal{D}) \leq \theta(\sigma_1', \mathcal{D})$$

     where $p_1 = (\lfloor h/m_\tau(\mathcal{R}) \rfloor + 1), h = \max(\varphi(\mathcal{C}_1(X), N_t - \tau(\sigma_{\mathcal{R}}))$. From Definition 4, $\varphi(\mathcal{C}(X)) = \varphi(\mathcal{C}_1(X))$, which implies $p \geq p_1$. By Definition 1, $\sigma_1' \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p} \mathcal{R}^j))$. Let $\sigma' = \sigma_1' \hat{} \sigma_{\mathcal{R}}$. Then, by Definition 1, $\sigma' \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p} \mathcal{R}^j))$. Furthermore, $\tau(\sigma') \geq N_t$ and $\theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$, the claim follows.
  2. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

     $$\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*) \oplus \mathcal{R}_1 \;\text{ and }\; \mathcal{C}(\oplus_{j=0}^{p} \mathcal{R}^j) = \mathcal{C}_1(\oplus_{j=0}^{p} \mathcal{R}^j) \oplus \mathcal{R}_1 .$$

By Definition 3, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$. By the assumption of the lemma, $\omega(\mathcal{C}(X)) = \infty$, which implies $\omega(\mathcal{C}_1(X)) = \infty$. By Definition 1,

$$\mathcal{L}(\mathcal{C}(\mathcal{A}^*)) = \mathcal{L}(\mathcal{C}_1(\mathcal{A}^*)) \cup \mathcal{L}(\mathcal{R}), \text{ and}$$
$$\mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{A}^j)) = \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{A}^j)) \cup \mathcal{L}(\mathcal{R}),$$

from which and from the inductive hypothesis, the result follows immediately.

3. Let $\mathcal{C}(X) = \mathcal{C}_1(X)^*$. Then

$$\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*))^* \text{ and} \mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))^*.$$

By Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$, from which and from the assumption of the lemma, we have $\omega(\mathcal{C}_1(X)) = \infty$. By Definition 1, for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, $\sigma = \sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m$ where for each i $(1 \leq i \leq m)$, $(\sigma_i \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$. Thus, by the assumption,

$$\tau(\sigma) = \tau(\sigma_1) + \tau(\sigma_2) + \ldots + \tau(\sigma_m) \geq N_t.$$

Let $N_t = N_{t1}+N_{t2}+\ldots+N_{tm}$ such that $\tau(\sigma_i) \geq N_{ti}$ $(1 \leq i \leq m)$. By the inductive hypothesis, for each i $(1 \leq i \leq m)$, there is $\sigma_i' \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p_i}\mathcal{R}^j))$ such that

$$\tau(\sigma_i') \geq N_{ti} \text{ and } \theta(\sigma_i, \mathcal{D}) \leq \theta(\sigma_i', \mathcal{D})$$

where $p_i = (\lfloor h/m_\tau(\mathcal{A})\rfloor + 1), h = \max(\varphi(\mathcal{C}_1(X), N_{ti})$. From Definition 4, $\varphi(\mathcal{C}(X)) = \varphi(\mathcal{C}_1(X))$, which implies for each $i \leq m$, $p \geq p_i$. Therefore, for each $i \leq m$, $\sigma_i' \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))$. Let $\sigma' = \sigma_1'\hat{}\sigma_2'\hat{}\ldots\hat{}\sigma_m'$. Then obviously,

$$\tau(\sigma') \geq N_t, \quad \theta(\sigma, \mathcal{D}) \leq \theta(\sigma', \mathcal{D}), \quad \text{and} \quad \sigma' \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))^*),$$

i.e., the lemma holds.

4. Let $\mathcal{C}(X) = (\mathcal{C}_1(X), [a, b])$ where $a \in R^+, b \in R^+, b > 0$, and $a \leq b$. Then $\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*), [a, b])$ and $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a, b])$. By Definition 4, $\omega(\mathcal{C}(X)) = \min(\omega(\mathcal{C}_1(X)), b)$. Since, by the assumption of the lemma, $\omega(\mathcal{C}(X)) = \infty$, it follows that $\omega(\mathcal{C}_1(X)) = \infty$ and $b = \infty$. Let $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, by Definition 1, $\tau(\sigma) \geq a$ and $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$. By the inductive hypothesis, there is $\sigma' \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p_1}\mathcal{R}^j))$ such that

$$\tau(\sigma') \geq \max(N_t, a) \text{ and } \theta(\sigma_i, \mathcal{D}) \leq \theta(\sigma', \mathcal{D})$$

where $p_1 = (\lfloor h/m_\tau(\mathcal{R})\rfloor + 1), h = \max(\varphi(\mathcal{C}_1(X), \max(N_t, a))$. Since, by Definition 4, $\varphi(\mathcal{C}(X)) = \max(\varphi(\mathcal{C}_1(X)), a)$, it should be the case that $p = p_1$. Hence, because $\tau(\sigma') \geq a$, $\sigma' \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, i.e. the claim holds. □

Lemma 9. *(3.)* Suppose $\omega(\mathcal{C}(X)) = \infty$, and $\mathcal{R}$ be a nonzero-simple HRE such that $M_\theta(\mathcal{R}) > 0$. Then for any nonnegative real numbers N_t and M_r, there is $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ such that $\tau(\sigma) \geq N_t$ and $\theta(\sigma, \mathcal{D}) > M_r$.

Proof. We prove the claim by induction on the structure of context.

- **Basic Case:** Let $\mathcal{C}(X) = X$. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{R}^*$. Since $M_\theta(\mathcal{R}) > 0$, we can choose $\sigma' \in \mathcal{L}(\mathcal{R})$ such that $\theta(\sigma', \mathcal{D}) = M_\theta(\mathcal{R})$, and let $\sigma = \underbrace{\sigma'\hat{}\sigma'\hat{}\ldots\hat{}\sigma'}_{k}$, where $k = \max(\lfloor M_r/M_\theta(\mathcal{R})\rfloor + 1, \lfloor N_t/m_\tau(\mathcal{R})\rfloor + 1)$. From Definition 1, it follows that $\sigma \in \mathcal{L}(\mathcal{R}^*)$. By the definition of k and since $\tau(\sigma') \geq m_\tau(\mathcal{R})$, we have $\tau(\sigma) > N_t$ and $\theta(\sigma, \mathcal{D}) > M_r$, which means that the basic case holds.
- **Induction Step:** Assume that claim holds for a context $\mathcal{C}_1(X)$, and let $\mathcal{C}(X)$ be defined form $\mathcal{C}_1(X)$.
  1. Let $\mathcal{C}(X) = \mathcal{C}_1(X)\hat{}\mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*)\hat{}\mathcal{R}_1$. By Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$. Since, by the assumption of the lemma, $\omega(\mathcal{C}(X)) = \infty$, it follows $\omega(\mathcal{C}_1(X)) = \infty$. Let $\sigma_\mathcal{R} \in \mathcal{L}(\mathcal{R}_1)$. By the inductive hypothesis, there is $\sigma' \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$ such that $\tau(\sigma') > N_t$ and $\theta(\sigma', \mathcal{D}) > M_r + \mid \theta(\sigma_\mathcal{R}, \mathcal{D}) \mid$. Let $\sigma = \sigma'\hat{}\sigma_\mathcal{R}$. Then $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, $\tau(\sigma) > N_t$, and $\theta(\sigma, \mathcal{D}) > M_r$, i.e. the lemma holds.
  2. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*) \oplus \mathcal{R}_1$. By Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$, and the lemma follows immediately from the inductive hypothesis.
  3. Let $\mathcal{C}(X) = \mathcal{C}_1(X)^*$. Then $\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*))^*$. Since by Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$, and since, by the assumption, $\omega(\mathcal{C}(X)) = \infty$, it holds that $\omega(\mathcal{C}_1(X)) = \infty$. By the inductive hypothesis, there is $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$ such that $\tau(\sigma) \geq N_t$ and $\theta(\sigma, \mathcal{D}) > M_r$. Since, by Definition 1, $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, the lemma holds.
  4. Let $\mathcal{C}(X) = (\mathcal{C}_1(X), [a, b])$ where $a \in R^+, b \in R^+, b > 0$, and $a \leq b$. Then $\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*), [a, b])$. By Definition 4, $\omega(\mathcal{C}(X)) = min(\omega(\mathcal{C}_1(X)), b)$. By the assumption of the lemma, $\omega(\mathcal{C}(X)) = \infty$, which implies $b = \infty$ and $\omega(\mathcal{C}_1(X)) = \infty$. By the inductive hypothesis, there is $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$ such that $\tau(\sigma) > max(N_t, a)$ and $\theta(\sigma, \mathcal{D}) > M_r$. Because $\tau(\sigma) \geq a$, by Definition 1, $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$. This completes the proof. □

Lemma 10. *(4.)* Suppose $\mathcal{R}$ be a nonzero-simple HRE, and $T \neq \infty$. Then for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, $\tau(\sigma) \leq T$ implies $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, where $p = \lfloor T/m_\tau(\mathcal{R})\rfloor + 1$.

Proof. We prove the claim by induction on the structure of context.

- **Basic Case:** Let $\mathcal{C}(X) = X$. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{R}^*$ and $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = \oplus_{j=0}^{p}\mathcal{R}^j$. Let $\sigma \in \mathcal{L}(\mathcal{R}^*)$ and $\tau(\sigma) \leq T$. By Definition 1, $\sigma = \sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m$ where for each i $(1 \leq i \leq m)$, $\sigma_i \in \mathcal{L}(\mathcal{R})$. Since

$$\tau(\sigma) = \tau(\sigma_1) + \tau(\sigma_2) + \ldots + \tau(\sigma_m) \leq T\,,$$

  it holds that $m \leq p$. By Definition 1, $\sigma \in \mathcal{L}(\oplus_{j=0}^{p}\mathcal{A}^j)$, i.e. the basic case holds.
- **Induction Step:** Assume that the claim holds for a context $\mathcal{C}_1(X)$, and let $\mathcal{C}(X)$ be defined from $\mathcal{C}_1(X)$.

1. Let $\mathcal{C}(X) = \mathcal{C}_1(X)\hat{}\mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

$$\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*)\hat{}\mathcal{R} \text{ and } \mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = \mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)\hat{}\mathcal{R}\,.$$

By Definition 1, any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ is of the form $\sigma_1\hat{}\sigma_{\mathcal{R}}$ where

$$\sigma_1 \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*)) \text{ and } \sigma_{\mathcal{R}} \in \mathcal{L}(\mathcal{R}_1))\,.$$

Since $\tau(\sigma) \leq T$, it follows that $\tau(\sigma_1) \leq T$. By the inductive hypothesis, we have $\sigma_1 \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))$. By Definition 1, $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, i.e. the claim holds.

2. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

$$\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*) \oplus \mathcal{R}_1 \text{ and } \mathcal{C}(\oplus_{j=0}^{p}\mathcal{A}^j) = \mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{A}^j) \oplus \mathcal{R}_1\,.$$

By Definition 1, we have

$$\begin{array}{l}\mathcal{L}(\mathcal{C}(\mathcal{R}^*)) = \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*)) \cup \mathcal{L}(\mathcal{R}_1), \text{and}\\ \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j)) = \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)) \cup \mathcal{L}(\mathcal{R}_1)\,.\end{array}$$

For any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, if $\sigma \in \mathcal{L}(\mathcal{R}_1)$ then $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, i.e. the claim holds; if $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$ then by the inductive hypothesis, the claim holds.

3. Let $\mathcal{C}(X) = \mathcal{C}_1(X)^*$. Then

$$\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*))^* \text{ and } \mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))^*\,.$$

By Definition 1, any $\sigma \in \mathcal{L}(\mathcal{R}^*)$ is of the form $\sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m$ where for each i $(1 \leq i \leq m)$, $\sigma_i \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$. Since $\tau(\sigma) \leq T$, for each i $(1 \leq i \leq m)$, $\tau(\sigma_i) \leq T$. By the inductive hypothesis, it follows that $\sigma_i \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))$ $(1 \leq i \leq m)$. From Definition 1, it follows that $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, i.e. the claim holds.

4. Let $\mathcal{C}(X) = (\mathcal{C}_1(X), [a, b])$ where $a \in R^+, b \in R^+ \cup \{\infty\}, b > 0$, and $a \leq b$. Then

$$\mathcal{C}(\mathcal{R}^*) = ((\mathcal{C}_1(\mathcal{R}^*)), [a, b]) \text{ and } \mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = ((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)), [a, b])\,.$$

For any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ such that $\tau(\sigma) \leq T$, by Definition 1, we have $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$. By the inductive hypothesis, $\sigma \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))$. Since $a \leq \tau(\sigma) \leq b$, $\sigma \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a, b]))$. By Definition 1,

$$\sigma \in \mathcal{L}((\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a, b]))\,,$$

i.e. the claim holds. □

Lemma 11. *(5.)*Suppose $\mathcal{R}$ be a nonzero-simple HRE, $a \in R^+$, $b \in R^+ \cup \{\infty\}, a \leq b$, and $b > 0$. Then

$$\mathcal{L}((\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a, b])) \supseteq \mathcal{L}((\mathcal{C}(\mathcal{R}^*), [a, b]))\,,$$

where $p = \lfloor b/m_\tau(\mathcal{R}) \rfloor + 1$.

Proof. We prove the claim by induction on the structure of context.

- **Basic Case:** Let $\mathcal{C}(X) = X$. Then

$$(\mathcal{C}(\mathcal{R}^*), [a,b]) = (\mathcal{R}^*, [a,b]) \text{ and } (\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]) = (\oplus_{j=0}^{p}\mathcal{R}^j, [a,b])\,.$$

  By Definition 1, any $\sigma \in \mathcal{L}((\mathcal{C}(\mathcal{R}^*), [a,b]))$ is of the form $\sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m$ where for each i $(1 \leq i \leq m)$, $\sigma_i \in \mathcal{L}(\mathcal{R})$. Since, by Definition 1, $a \leq \tau(\sigma) \leq b$, $a \leq \tau(\sigma_1) + \tau(\sigma_2) + \ldots + \tau(\sigma_m) \leq b$. It follows that $m \leq p$. By Definition 1, $\sigma \in \mathcal{L}((\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]))$, which implies that the basic case holds.
- **Induction Step:** Assume that the claim holds for a context $\mathcal{C}_1(X)$, and let $\mathcal{C}(X)$ be defined from $\mathcal{C}_1(X)$
  1. Let $\mathcal{C}(X) = \mathcal{C}_1(X)\hat{}\mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

$$\begin{array}{l}(\mathcal{C}(\mathcal{R}^*), [a,b]) = (\mathcal{C}_1(\mathcal{R}^*)\hat{}\mathcal{R}_1, [a,b]), \text{ and}\\ (\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)\hat{}\mathcal{R}_1, [a,b])\,.\end{array}$$

     From Definition 1, it follows that any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ is of the form $\sigma_1\hat{}\sigma_{\mathcal{R}}$ where $\sigma_1 \in \mathcal{L}((\mathcal{C}_1(\mathcal{R}^*), [a,b]))$ and $\sigma_{\mathcal{R}} \in \mathcal{L}(\mathcal{R}_1)$. By the inductive hypothesis, $\sigma_1 \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]))$. By Definition 1, $\sigma \in \mathcal{L}((\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]))$, which implies the claim.
  2. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then

$$\begin{array}{l}(\mathcal{C}(\mathcal{R}^*), [a,b]) = (\mathcal{C}_1(\mathcal{R}^*) \oplus \mathcal{R}_1, [a,b]), \text{ and}\\ (\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j) \oplus \mathcal{R}_1, [a,b])\,.\end{array}$$

     From Definition 1, it follows that

$$\begin{array}{l}\mathcal{L}((\mathcal{C}(\mathcal{R}^*), [a,b])) = \mathcal{L}((\mathcal{C}_1(\mathcal{R}^*), [a,b])) \cup \mathcal{L}(\mathcal{R}_1), \text{ and}\\ \mathcal{L}((\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b])) = \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b])) \cup \mathcal{L}(\mathcal{R}_1)\,.\end{array}$$

     By the inductive hypothesis, the claim holds.
  3. Let $\mathcal{C}(X) = \mathcal{C}_1(X)^*$. Then

$$\begin{array}{l}(\mathcal{C}(\mathcal{R}^*), [a,b]) = (\mathcal{C}_1(\mathcal{R}^*)^*, [a,b]), \text{ and}\\ (\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)^*, [a,b])\,.\end{array}$$

     By Definition 1, any $\sigma \in \mathcal{L}((\mathcal{C}(\mathcal{R}^*), [a,b]))$ has the form $\sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m$ where for each $\sigma_i \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$ $(1 \leq i \leq m)$. By Definition 1, for each i $(1 \leq i \leq m)$, $\sigma_i \in \mathcal{L}((\mathcal{C}_1(\mathcal{R}^*), [0,b]))$. From the inductive hypothesis, it follows that $\sigma_i \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [0,b]))$ $(1 \leq i \leq m)$ and $\sigma \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [0,b])^*)$. Since $a \leq \tau(\sigma) \leq b$, by Definition 1, $\sigma \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)^*, [a,b]))$, which implies the claim.
  4. Let $\mathcal{C}(X) = (\mathcal{C}_1(X), [a_1,b_1])$ where $a_1 \in R^+, b_1 \in R^+, b_1 > 0$, and $a_1 \leq b_1$. Then

$$\begin{array}{l}(\mathcal{C}(\mathcal{R}^*), [a,b]) = ((\mathcal{C}_1(\mathcal{R}^*), [a_1,b_1]), [a,b]), \text{ and}\\ (\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j), [a,b]) = ((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a_1,b_1]), [a,b])\,.\end{array}$$

For any $\sigma \in \mathcal{L}((\mathcal{C}(\mathcal{R}^*), [a, b]))$, by Definition 1, $\sigma \in \mathcal{L}((\mathcal{C}_1(\mathcal{R}^*), [a_1, b_1]))$. By the inductive hypothesis, $\sigma \in \mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a_1, b_1]))$. Since $a \leq \tau(\sigma) \leq b$, by Definition 1, $\sigma \in \mathcal{L}(((\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a_1, b_1]), [a, b]))$, i.e. the claim holds. □

Lemma 12. *(6.)* Let $\mathcal{R}$ be a nonzero-simple HRE. Then $\mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j)) \supseteq \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, where $p = \lfloor \omega(\mathcal{C}(X))/m_\tau(\mathcal{R}) \rfloor + 1$.

Proof. The proof goes by induction on the structure of context.

- **Basic Case:** Let $\mathcal{C}(X) = X$. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{R}^*$ and $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = \oplus_{j=0}^{p}\mathcal{R}^j$. By Definition 4, $\omega(\mathcal{C}(X)) = \infty$. It follows that $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = \oplus_{j=0}^{\infty}\mathcal{R}^j$. Since, by Definition 1, $\mathcal{L}(\mathcal{R}^*) = \mathcal{L}(\oplus_{j=0}^{\infty}\mathcal{R}^j)$, the basic case holds.
- **Induction Step:** Assume that the claim holds for a context $\mathcal{C}_1(X)$, and let $\mathcal{C}(X)$ be defined from $\mathcal{C}_1(X)$.
  1. Let $\mathcal{C}(X) = \mathcal{C}_1(X)\hat{}\mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then $\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*)\hat{}\mathcal{R}_1$ and $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = \mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)\hat{}\mathcal{R}_1$. By Definition 1, any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ is of the form $\sigma_1\hat{}\sigma_{\mathcal{R}}$ where $\sigma_1 \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$ and $\sigma_{\mathcal{R}} \in \mathcal{L}(\mathcal{R}_1)$. Since, by Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$, by the inductive hypothesis, $\sigma_1 \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))$. It follows that $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, which implies the claim.
  2. Let $\mathcal{C}(X) = \mathcal{C}_1(X) \oplus \mathcal{R}_1$ where $\mathcal{R}_1$ is an HRE. Then
  $$\mathcal{C}(\mathcal{R}^*) = \mathcal{C}_1(\mathcal{R}^*) \oplus \mathcal{R}_1 \quad \text{and} \quad \mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = \mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j) \oplus \mathcal{R}_1 \,.$$
  By Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$. Since by Definition 1,
  $$\begin{aligned} &\mathcal{L}(\mathcal{C}(\mathcal{R}^*)) = \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*)) \cup \mathcal{L}(\mathcal{R}_1), \text{ and} \\ &\mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j)) = \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j)) \cup \mathcal{L}(\mathcal{R}_1)\,, \end{aligned}$$
  for any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$, if $\sigma \in \mathcal{L}(\mathcal{R}_1)$, then obviously the claim holds; if $\sigma \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))$, then by the inductive hypothesis, the claim holds.
  3. Let $\mathcal{C}(X) = \mathcal{C}_1(X)^*$. Then
  $$\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*))^* \text{ and } \mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))^* \,.$$
  By Definition 4, $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$. By Definition 1, any $\sigma \in \mathcal{L}(\mathcal{C}(\mathcal{R}^*))$ has the form $\sigma_1\hat{}\sigma_2\hat{}\ldots\hat{}\sigma_m$ $(\sigma_i \in \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*)), 1 \leq i \leq m)$. From the inductive hypothesis, it follows that each $\sigma_i \in \mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j))$ $(1 \leq i \leq m)$. By Definition 1, $\sigma \in \mathcal{L}(\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j))$, which implies the claim.
  4. Let $\mathcal{C}(X) = (\mathcal{C}_1(X), [a, b])$ where $a \in R^+, b \in R^+ \cup \{\infty\}, b > 0$, and $a \leq b$. Then $\mathcal{C}(\mathcal{R}^*) = (\mathcal{C}_1(\mathcal{R}^*), [a, b])$ and $\mathcal{C}(\oplus_{j=0}^{p}\mathcal{R}^j) = (\mathcal{C}_1(\oplus_{j=0}^{p}\mathcal{R}^j), [a, b])$. By the inductive hypothesis,
  $$\mathcal{L}(\mathcal{C}_1(\oplus_{j=0}^{p_1}\mathcal{R}^j)) \supseteq \mathcal{L}(\mathcal{C}_1(\mathcal{R}^*))\,,$$
  where $p_1 = \lfloor \omega(\mathcal{C}_1(X))/m_\tau(\mathcal{R}) \rfloor + 1$. From Definition 1, it follows that
  $$\mathcal{L}((\mathcal{C}_1(\oplus_{j=0}^{p_1}\mathcal{R}^j), [a, b])) \supseteq \mathcal{L}((\mathcal{C}_1(\mathcal{R}^*), [a, b]))\,.$$
  By Definition 4, $\omega(\mathcal{C}(X)) = min(\omega(\mathcal{C}_1(X)), b)$. If $\omega(\mathcal{C}(X)) = \omega(\mathcal{C}_1(X))$, since $p = p_1$, the claim follows immediately; if $\omega(\mathcal{C}(X)) = b$, by Lemma 5, the claim holds; □